TECHNOLOGY DEPT.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 61 NUMBER 9

FOR LIBRARY

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NOVEMBER 1954

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Carl B. Allendoerfer, *Editor*

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THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

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Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

AN ELEMENTARY ANALOGUE TO THE GAUSS-BONNET THEOREM

G. PÓLYA, Stanford University

1. We consider a polyhedral surface P of the topological type of a circular disk, bounded by a (skew) closed polygon R (the rim) without double points. We shall appropriately define S, the *area of the spherical image* of P, and T, *the total change of direction* of R on P, and prove that

$$S + T = 2\pi.$$

This is an elementary analogue to, or an elementary limiting case of, the Gauss-Bonnet theorem which itself can be considered as a limiting case of our elementary theorem (when the faces become infinitely small and their number infinitely large).

Let F, E, and V denote the number of faces, edges, and vertices of P, respectively. Since P is a (simply connected) open polyhedral surface, Euler's well-known theorem assumes here the form

$$F - E + V = 1.$$

Euler's theorem is closely connected with our analogue of the Gauss-Bonnet theorem. In fact, we shall kill two birds with one stone, and derive both theorems together.

2. Our proof needs two well-known lemmas and the introduction of appropriate notation.

LEMMA I. *The sum of the angles in a plane polygon with n sides is* $(n-2)\pi$.

LEMMA II. *We consider the polyhedral angle H and its polar polyhedral angle H′. We call L the sum of the face angles of H, and A′ the measure of the solid angle included by H′. Then*

$$A' = 2\pi - L.$$

According to the well-known definition of the polar polyhedral angles, each edge of H' corresponds, and is perpendicular to, a face of H. If we assume for a moment that H and H' have the same vertex and describe a unit sphere about this common vertex as center, each polyhedral angle intersects the sphere in a spherical polygon: the spherical polygon associated with H has the perimeter L and that associated with H' the area A'.

Let F_3, F_4, F_5, $\cdots$ denote the number of the triangular, quadrilateral, pentagonal, $\cdots$ faces of P, respectively. Obviously

$$F_3 + F_4 + F_5 + \cdots = F. \tag{1}$$

Some of the edges of P do not, and others do, belong to the boundary R of P. Let E_i denote the number of the former (interior) edges, and E_b that of the latter (boundary) edges. Obviously

$$E_i + E_b = E. \tag{2}$$

Some of the vertices of P do not, and others do, belong to the boundary R. Let V_i denote the number of the former, V_b that of the latter. Obviously

$$V_i + V_b = V. \tag{3}$$

Let α stand for any angle of any of the F faces of P, $\sum\alpha$ for the sum of all such angles, $\sum_i\alpha$ for the sum of those angles whose vertex does not belong to R, and $\sum_b\alpha$ for the sum of those whose vertex does belong to R. Obviously

$$\sum_i \alpha + \sum_b \alpha = \sum \alpha. \tag{4}$$

It is easily seen that

$$3F_3 + 4F_4 + 5F_5 + \cdots = 2E_i + E_b, \tag{5}$$

$$E_b = V_b, \tag{6}$$

$$\sum \alpha = \pi F_3 + 2\pi F_4 + 3\pi F_5 + \cdots. \tag{7}$$

In deriving (7), we use Lemma I.

3. The total change of direction T of the boundary R of P is the sum of the changes of direction of R at its several vertices. If one of these vertices lies on the boundary of just one of the F faces of P, and the angle of that face at that vertex is α, the change of direction is $\pi-\alpha$. If, however, one of the vertices of R belongs to several contiguous faces of P and these faces have at that vertex the angles α', α'', α''', $\cdots$ respectively, the change of direction is $\pi-\alpha'-\alpha''-\alpha'''-\cdots$. Therefore, the total change of direction is

$$T = \pi V_b - \sum_b \alpha. \tag{8}$$

4. In taking the next step, we restrict ourselves to the most intuitive case. We assume that we started from a convex polyhedron C, drew the not self-intersecting closed polygonal line R on its surface, and obtained P as one of the two portions into which R divides the surface of C. We draw to each of the faces of P a normal (an *outer* normal to C). Then we take a unit sphere about a fixed point O of space, and draw to each normal a parallel radius. These radii intersect the surface of the unit sphere in F points; we shall use these F points in constructing the spherical image of P. In fact, let H be the polyhedral angle (interior to C) included by those faces of P that meet in one of the V_i interior vertices of P. Let us draw H', the polar polyhedral angle to H, with O as vertex. The vertices of the spherical polygon associated with H' are among the F points that we have just constructed. The full spherical image of P consists of such spherical polygons, each corresponding to an interior vertex of P. Computing the area of each of these V_i polygons on the basis of Lemma II, we find that the total area of the spherical image is

$$S = 2\pi V_i - \sum_i \alpha. \tag{9}$$

5. We use first (9), (8), and (4), and we pass to the following lines by using (7), then (5) and (1), and finally (6):

$$\begin{aligned} S + T &= 2\pi V_i + \pi V_b - \sum \alpha \\ &= \pi[2V_i + V_b - (F_3 + 2F_4 + 3F_5 + \cdots)] \\ &= \pi[2V_i + V_b - (2E_i + E_b - 2F)] \\ &= \pi[2F - 2E_i - 2E_b + 2V_i + 2V_b]. \end{aligned}$$

Using (2) and (3) we obtain:

$$S + T = 2\pi[F - E + V]. \tag{10}$$

Now let us observe an important point. There is a "flat" polyhedral surface P_f that has exactly F faces, E edges, and V vertices, all contained in the same plane. (If we have obtained P from a convex polyhedron C by dividing the surface of C into two portions by the closed line R, let us choose any point p on the "other side" of R; that is, p belongs to the surface of C, but does not belong to P or R. Now, P is "fully visible" from p; that is, we can project P from p as center of projection onto a plane in one-one fashion: the projection so obtained can be taken as P_f, which is, in fact, the division of a plane polygon with E_b sides into F polygons. Obviously, we could flatten P into P_f by and by, that is, pass from one to the other by a continuous deformation.) Our foregoing derivation and its result (10) are valid not only for P, but also for P_f. Yet, obviously, the spherical image of P_f reduces to a point, its area S_f is nil, and T_f the total change of direction of the boundary of P_f, is 2π (by Lemma I), and so the left hand side of (10) reduces to $0+2\pi$: yet the right hand side is the same for P_f as for P. Therefore, we obtain at one stroke

$$S + T = 2\pi, \qquad F - E + V = 1.$$

6. In order to apply the variant just proved of the Gauss-Bonnet theorem, let us consider each of the F faces of P as a rigid plate; these plates are joined along the E_i interior edges as with hinges. If some of the V_i interior vertices have more than three edges, the polyhedral surface P, although consisting of rigid plates, may be flexible, of variable shape. When P varies, also the shape of its spherical image may vary. Yet T, the total change of direction along the rim of P, remains unchanged, and so must remain also the *area* $S=2\pi-T$ of the spherical image. This is the elementary counterpart of the ***theorema egregium*** of Gauss according to which the (Gaussian) curvature of a surface remains unchanged when the surface is bent.*

The scope of the theorem proved would be considerably widened by a clear discussion of the validity of Lemma II for non-convex polyhedral angles.

* *Cf.* D. Hilbert and S. Cohn-Vossen, Anschauliche Geometrie, pp. 172–173.

CONCERNING THE COMPLETE QUADRILATERAL*

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1. Notation. Let A', B', C' be the points in which a transversal Δ cuts the sides BC, CA, AB of a triangle $T \equiv ABC$; the triangle and the transversal determine the complete quadrilateral $Q \equiv (ABC, \Delta)$. Let H, H_a, H_b, H_c and O, O_a, O_b, O_c denote the orthocenters and the centers of the circumcircles (O), (O_a), (O_b), (O_c) of triangles T, $T_a \equiv AB'C'$, $T_b \equiv BC'A'$, $T_c \equiv CA'B'$.

2. Circles through the Miquel point. The lines MA', MB', MC' which join the point of concurrency M of the circles (O), (O_a), (O_b), (O_c) (the Miquel point of quadrilateral Q) to the points of intersection of Δ with the lines BC, CA, AB meet these lines in the same angle θ (the Miquel angle of Q). The rotation with center M and angle θ transforms the lines MA, MB, MC into the tangents through M to the circles (O_a), (O_b), (O_c). The rotation with center M and angle $\pi/2-\theta$ (mod π) then transforms the rays MA, MB, MC into the rays MO_a, MO_b, MO_c and, since

$$MA/MO_a = MB/MO_b = MC/MO_c = 2 \sin \theta,$$

triangles T and $O_aO_bO_c$ correspond under the similitude having center M, angle $\pi/2-\theta$ (mod π), and ratio $2 \sin \theta$. Similarly, triangles T_a and OO_bO_c, T_b and OO_cO_a, T_c and OO_aO_b correspond under similitudes having centers at M and ratios $2 \sin \theta_a$, $2 \sin \theta_b$, $2 \sin \theta_c$, θ_a, θ_b, θ_c being the angles in which lines MA, MB, MC cut the transversal Δ.

The lines A_1A, A_1B, A_1C joining an arbitrary point A_1 of circle (O) to the vertices of T recut the circles (O_a), (O_b), (O_c), respectively, in A_1', B_1', C_1'. Since circles (O), (O_b), (O_c) are concurrent in M, the points A', B_1', C_1' are collinear and circle $A_1B_1'C_1'$ passes through point M. Similarly, B', C_1', A_1' and C', A_1', B_1' are sets of collinear points, and circles $A_1C_1'A_1'$ and $A_1A_1'B_1'$ pass through M. It follows that circles $A_1B_1'C_1'M$, $A_1C_1'A_1'M$, $A_1A_1'B_1'M$ coincide with circle $(O_1) \equiv A_1'B_1'C_1'$, passing through M. These remarks can be successively extended to arbitrary points $A_2, A_3, \cdots, A_n$ of circle (O) circumscribed about triangle T, to which will correspond circles (O_2), (O_3), $\cdots$, (O_n) circumscribed about triangles $A_2'B_2'C_2'$, $A_3'B_3'C_3'$, $\cdots$, $A_n'B_n'C_n'$ and all concurring in the Miquel point M of quadrilateral Q. Finally, one obtains analogous results by taking arbitrary points on circles (O_a), (O_b), (O_c) and joining them to the vertices of triangles T_a, T_b, T_c, so that in addition to the circles (O), (O_a), (O_b), (O_c) passing through M there are four infinite sets of circles passing through M, like the set (O), (O_1), (O_2), $\cdots$.

It follows from the similitude having center M and ratio $2 \sin \theta_a$, set up by triangles T_a and OO_bO_c, that the isosceles triangles MOA, MO_bC', MO_cB' are similar. On the other hand, it is clear that the complete quadrilaterals $Q_a \equiv (T_a, BC)$ and $(AC'A_1', BA_1)$ have the same Miquel circle and Miquel angle

* Translated from the French by Howard Eves.

$$\theta_a = (BM, BA) = (A_1M, A_1A).$$

Triangles T_a and OO_bO_c, $AA_1'B'$ and OO_aO_1, and consequently quadrangles $AC'B'A_1'$ and $OO_bO_cO_1$, then correspond under a similitude having center M and ratio $2 \sin \theta_a = 2 \cos (\pi/2 - \theta_a)$. In exactly the same way one arrives at the same conclusion for the quadrangles $AC'B'A_2'$ and $OO_bO_cO_2$, $\cdots$, $AC'B'A_n'$ and $OO_bO_cO_n$, and, finally, for the polygons $AC'B'A_1'A_2' \cdots A_n'$ and $OO_bO_cO_1 \cdots O_n$.

CONSEQUENCE. *Polygon $OO_bO_cO_1O_2 \cdots O_n$ is inscribed in the Miquel circle $(\omega) \equiv OO_aO_bO_cM$ of quadrilateral Q.*

THEOREM. *The four sets of circles defined above envelop a cardioid (C) with cusp at M, and which coincides with the transform, under the homothety $(M, 2)$, of the pedal of M with respect to circle (ω).*

The proof follows immediately from the preceding *consequence* and from the property which the circles (O_1), (O_2), $\cdots$, (O_n) have of passing through the fixed point M on circle (ω).

COROLLARY [1]. *The circles (O), (O_a), (O_b), (O_c) also envelop the cardioid (C).*

For if point A_1 coincides with vertex A of triangle T, for example, then line A_1A coincides with the tangent to circle (O) at A, points B_1', C_1' coincide with C', B', and circles $(O_1) \equiv A_1'B'C'$ and $(O_a) \equiv AB'C'$ coincide. Similar conclusions hold for circles (O), (O_b), (O_c).

3. A generalization. If, in the preceding, one considers three *non-collinear* points A', B', C', taken arbitrarily on the sides BC, CA, AB of triangle T, certain of the properties which we have obtained continue to hold, for the circles $(O_a) \equiv AB'C'$, $(O_b) \equiv BC'A'$, $(O_c) \equiv CA'B'$ still concur in a point M. On the other hand [2], lines A_1A, A_1B, A_1C joining an arbitrary point A_1 of the plane to the vertices of triangle T recut circles (O_a), (O_b), (O_c) in the vertices A_1', B_1', C_1' of a triangle of which the circumcircle passes through M. If the point A_1 describes circle $(O) \equiv ABC$, the center O_1 of circle $A_1'B_1'C_1'$ describes circle (ω) circumscribed about triangle $O_aO_bO_c$, because of the similarity of triangles T and $O_aO_bO_c$, A_1BC and $O_1O_bO_c$, A_1CA and $O_1O_cO_a$, A_1AB and $O_1O_aO_b$.

CONSEQUENCE. *As point A_1 varies on circle (O), circle $(O_1) \equiv A_1'B_1'C_1'$ passes through the fixed point M and envelops a limaçon of Pascal of pole M and generating circle (ω).*

4. Orthocenter of a polygon. If O, G, H denote the circumcenter, centroid, and orthocenter of a triangle T, we have the vector relation $3\vec{OG} = \vec{OH}$. More generally, if Ω represents the mean center of the n vertices A_1, A_2, $\cdots$, A_n of a polygon P inscribed in a circle (O) of center O, and Λ is the point such that

$$n\vec{O\Omega} = \vec{O\Lambda},$$

we shall call Λ the *orthocenter* of P.

THEOREM. *The perpendicular bisectors of the segments* OH_1, O_aH_a, O_bH_b, O_cH_c *joining the centers of circles* (O), (O_a), (O_b), (O_c) *to the orthocenters* H_1, H_a, H_b, H_c *of quadrangles* $ABCA_1$, $AB'C'A_1'$, $BC'A'B_1'$, $CA'B'C_1'$ *concur at the orthocenter* Λ *of the cyclic polygon* $OO_aO_bO_cO_1$.

Under the similitude of center M, angle $\pi/2-\theta_a \pmod{\pi}$, and ratio $2 \sin \theta_a$, determined by quadrangles $AC'A_1'B'$ and $OO_bO_1O_c$, for example, line O_aH_a intersects line ωh_1, joining the center of circle (ω) to the orthocenter h_1 of quadrangle $OO_bO_1O_c$, in an angle equal to $\pi/2-\theta_a \pmod{\pi}$. On the other hand, the resultant of the vectors $\overrightarrow{\omega O_a}$ and $\overrightarrow{\omega h_1}$ coincides with the vector $\overrightarrow{\omega\Lambda}$, for the quadrangle $\omega h_1 \Lambda O_a$ is a parallelogram, inasmuch as

$$\omega\Omega_1/\omega h_1 = 1/4 \quad \text{and} \quad \Omega_1\Omega/\Omega_1 O_a = 1/5,$$

where Ω_1 and Ω designate the mean centers of O, O_b, O_1, O_c and O, O_a, O_b, O_c, O_1, respectively. One then obtains

$$O_aH_a/2O_a\Lambda = \cos(\pi/2 - \theta_a) = \sin\theta_a,$$

which proves that the perpendicular bisector of segment O_aH_a, and similarly those of segments OH_1, O_bH_b, O_cH_c, passes through the orthocenter Λ of polygon $OO_aO_bO_cO_1$ inscribed in the Miquel circle (ω) of quadrilateral Q.*

THEOREM. *The perpendicular bisectors of the segments joining the centers* O, O_a, O_b, O_c *of circles* (O), (O_a), (O_b), (O_c) *to the orthocenters* H_1, H_a, H_b, H_c *of polygons* $ABCA_1A_2 \cdots A_n$, $AB'C'A_1'A_2' \cdots A_n'$, $BC'A'B_1'B_2' \cdots B_n'$, $CA'B'C_1'C_2' \cdots C_n'$ *concur at the orthocenter* Λ *of polygon* $OO_aO_bO_cO_1O_2 \cdots O_n$ *inscribed in the Miquel circle* (ω) *of quadrilateral* Q.

In order to establish this theorem it suffices to extend step by step the reasoning which we employed in the proof of the preceding theorem, which is a particular case of this one.

COROLLARY [3]. *The perpendicular bisectors of the segments* OH, O_aH_a, O_bH_b, O_cH_c *joining the circumcenters to the orthocenters of triangles* T, T_a, T_b, T_c *concur at the orthocenter* Λ *of the cyclic quadrangles* $OO_aO_bO_c$ (the Kantor-Hervey point).

For the center O_1 of circle $AB'C'A_1'$, for example, coincides with the center of circle (O_a) when point A_1 coincides with vertex A of triangle T.

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1. *Cf.* R. Deaux, Mathesis, 1925, p. 56.

2. A. Mannheim, Nouvelles Annales de Mathématiques, 1890, p. 239, question 1594; generalization by L. Bosi, p. 556; V. Thébault, Annales de la Société Scientifique de Bruxelles, 1937, pp. 16–22; this MONTHLY, 1947, pp. 420–421, problem 3895 (extension to a plane or a skew polygon of n vertices).

3. Kantor, Bulletin des Sciences Mathématiques et Astronomiques, 1879, p. 138; Hervey, Educational Times, Reprints, 1891, p. 37.

* This theorem, which is perhaps new, was communicated by the author in January, 1953, to M. R. Blanchard.

TWISTED CURVES AND THE MEAN-VALUE PROPOSITION

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1. Introduction. It is well known that the mean-value proposition is true for differentiable plane curves; that is, between any pair of points of a plane differentiable curve there exists a third point at which the tangent is parallel to the chord joining the pair of points. Now for twisted curves no such conclusion can be reached since, for example, twisted cubics and circular helices have no tangents parallel to any chord. Since any twisted sufficiently differentiable curve is locally approximately a twisted cubic, one might expect that the existence of a parallel tangent for each chord implies that the curve is plane.

In what follows a general space curve is represented by a vector $\mathbf{x}=\mathbf{x}(t)$ whose components are given by

$$x^i = x^i(t), \qquad i = 1, 2, 3,$$

where the x^i are rectangular cartesian coordinates in three-dimensional Euclidean space, and the functions $x^i(t)$ are real and single-valued on D, a real domain $a \leqq t \leqq b$. To say that $\mathbf{x}(t)$ belongs to class C^n means that the functions $x^i(t)$ have continuous derivatives of order n.

2. Space curves for which a mean-value proposition in the small is true.

THEOREM. *If*

(1) $\mathbf{x}(t)$ *belongs to* C^2,

(2) *the cross product* $\mathbf{x}' \times \mathbf{x}''$ *does not vanish for any value of* t *in* D,

(3) *for each* t *in* D *there exists a positive* ϵ *such that for* t_1 *and* t_2 *in* D *with* $t_1 < t_2$ *and* $|t_i - t| < \epsilon$ *there exist real numbers* τ *and* λ *for which*

$$\mathbf{x}(t_2) - \mathbf{x}(t_1) = \lambda \mathbf{x}'(\tau) \quad \text{and} \quad t_1 < \tau < t_2,$$

then $\mathbf{x}(t)$ *represents a plane curve.*

Condition (3), the mean-value proposition in the small, simply states that for any pair of points of the curve sufficiently close together there is a point between them on the curve at which the tangent is parallel to their chord.

The theorem is proved by use of the Taylor formulas

$$\mathbf{x}(t) = \mathbf{x}(0) + \mathbf{x}'(0)t + \mathbf{x}''(0)\frac{t^2}{2} + \mathbf{x}'''(0)\frac{t^3}{6} + \mathbf{a}(t)$$

and

$$\mathbf{x}'(t) = \mathbf{x}'(0) + \mathbf{x}''(0)t + \mathbf{x}'''(0)\frac{t^2}{2} + \mathbf{b}(t),$$

where

$$\lim_{t\to 0}\frac{\mathbf{a}(t)}{t^3}=0 \quad \text{and} \quad \lim_{t\to 0}\frac{\mathbf{b}(t)}{t^2}=0.$$

Assuming $\mathbf{x}(0)$ vanishes, we have from (3) that

$$\mathbf{x}(t)=\lambda\mathbf{x}'(\tau), \qquad 0<\tau<t<\epsilon, \tag{2.1}$$

or

$$\mathbf{x}'(0)t+\mathbf{x}''(0)\frac{t^2}{2}+\mathbf{x}'''(0)\frac{t^3}{6}+\mathbf{a}(t)=\lambda\left[\mathbf{x}'(0)+\mathbf{x}''(0)\tau+\mathbf{x}'''(0)\frac{\tau^2}{2}+\mathbf{b}(\tau)\right]. \tag{2.2}$$

Since (2) implies that $\mathbf{x}'(t)$ does not vanish for any value of t in D, it follows from (2.1) that

$$\lim_{t\to 0}\frac{\lambda}{t}=1.$$

The cross product of both members of (2.2) with $\mathbf{x}'(0)$ gives

$$\begin{aligned}&\mathbf{x}'(0)\times\mathbf{x}''(0)\frac{t^2}{2}+\mathbf{x}'(0)\times\mathbf{x}'''(0)\frac{t^3}{6}+\mathbf{x}'(0)\times\mathbf{a}(t)\\&\qquad=\lambda\left[\mathbf{x}'(0)\times\mathbf{x}''(0)\tau+\mathbf{x}'(0)\times\mathbf{x}'''(0)\frac{\tau^2}{2}+\mathbf{x}'(0)\times\mathbf{b}(\tau)\right].\end{aligned} \tag{2.3}$$

From condition (2), the fact that τ/t is bounded, and (2.3), we find that

$$\lim_{t\to 0}\frac{\tau}{t}=\frac{1}{2}.$$

The dot product of both members of (2.2) with $\mathbf{x}'(0)\times\mathbf{x}''(0)$ yields

$$\begin{aligned}&(\mathbf{x}'(0)\mathbf{x}''(0)\mathbf{x}'''(0))\frac{t^3}{6}+(\mathbf{x}'(0)\mathbf{x}''(0)\mathbf{a}(t))\\&\qquad=\lambda\left[(\mathbf{x}'(0)\mathbf{x}''(0)\mathbf{x}'''(0))\frac{\tau^2}{2}+(\mathbf{x}'(0)\mathbf{x}''(0)\mathbf{b}(\tau))\right].\end{aligned} \tag{2.4}$$

If we suppose that the triple scalar product $(\mathbf{x}'(0)\ \mathbf{x}''(0)\ \mathbf{x}'''(0))$ does not vanish, then from (2.4)

$$\lim_{t\to 0}\frac{\tau}{t}=\frac{\lambda}{\sqrt{3}}\neq\frac{1}{2},$$

a contradiction; therefore, $(\mathbf{x}'(0)\mathbf{x}''(0)\mathbf{x}'''(0))=0$. The fact that

$$(\mathbf{x}'(t)\mathbf{x}''(t)\mathbf{x}'''(t))\equiv 0 \tag{2.5}$$

is verified by repeating the above procedure after making the appropriate translation of coordinates and change of parameter t. From (1), (2), and (2.5) we conclude that $\mathbf{x}(t)$ represents a plane curve [1].

That condition (2) is needed is clear from the example of a curve consisting of the boundary of a surface formed by three mutually perpendicular faces of a cube with the portions near the vertices replaced by plane arcs sufficiently smooth to force the curve to satisfy (1). Such a curve satisfies conditions (1) and (3) but not (2). We also observe that for points far enough apart on this curve the chord has no parallel tangent. Actually if condition (3) is changed to include all chords, then (1) and (2) may be replaced by considerably weaker conditions.

3. Space curves for which a mean-value proposition in the large is true.

THEOREM. *If*

(1) $\mathbf{x}(t)$ belongs to C^1,

(2) $\mathbf{x}'(t)$ *does not vanish for any value of* t *in* D,

(3) *for every* t_1 *and* t_2 *in* D *there exist real numbers* τ *and* λ *for which* τ *is in* D *and*

$$\mathbf{x}(t_2) - \mathbf{x}(t_1) = \lambda \mathbf{x}'(\tau),$$

(4) $\mathbf{y} = \mathbf{x}'(t)/|\mathbf{x}'(t)|$ *for all* t *in* D *gives a subspace* T *of the unit sphere with dim* $T \leqq 1$,

then $\mathbf{x}(t)$ *represents a plane curve.*

Condition (3), the mean-value proposition in the large, simply states that any chord has a parallel tangent at some point of the curve, while condition (4) states that the spherical indicatrix T of tangents has dimension less than or equal to 1. The definition of dimension used here is the topological invariant defined in [2].

We show first that a curve satisfying (1) and (2) with no two of its tangent lines skew is a plane curve. If no two tangents are skew, then there are only three possibilities. The tangents all lie in the same plane (and the curve is a plane curve) or the tangents all pass through the same point or they are all parallel. In the last two cases with conditions (1) and (2) it can be demonstrated by examining the solutions of certain differential equations that the curve is a straight line segment and hence a plane curve.

If we suppose a curve which satisfies (1), (2), (3), and (4) is not a plane curve, then there exists a pair of points X_0, Y_0 of the curve at which the tangents $\underline{X_0X_0}$, $\underline{Y_0Y_0}$ to the curve are skew. Let arcs $\widehat{PQ}$ and $\widehat{RS}$ of the curve be small and disjoint, with $X_0 \in \text{int } \widehat{PQ}$ and $Y_0 \in \text{int } \widehat{RS}$. Let X be any point of $\widehat{PQ}$ and let X' be the point common to the line $\underline{X_0X_0}$ and the plane on X normal to $\underline{X_0X_0}$. Let Y be any point of $\widehat{RS}$ and let Y' be the point common to the line $\underline{Y_0Y_0}$ and the plane on Y normal to $\underline{Y_0Y_0}$. If XY denotes the point of the unit sphere in the direction from the center parallel to the segment $\overline{XY}$, then the set

H of all XY is a subset of the spherical indicatrix K of chords of the curve. The arcs $\widehat{PQ}$ and $\widehat{RS}$ are closely approximated by the segments $\overline{P'Q'}$ and $\overline{R'S'}$ of the tangents at X_0 and Y_0, since any tangent has a contact of order one (at least) with the curve. Furthermore, any point XY of H will be close to the corresponding point $X'Y'$. The set of all points $X'Y'$ is a subset H' of the unit sphere bounded by four great circle arcs with no interior points in common and with no three of the end points on a single great circle. The four curves on the unit sphere traversed by the points XR, PY, XS, QY, as X and Y traverse the arcs $\widehat{PQ}$ and $\widehat{RS}$, will be close to the four great circle arcs bounding H'. From continuity it is clear that H will cover completely a region bounded by the four curves and will closely approximate the region bounded by the four great circle arcs. Since dim H' is clearly 2, we obtain dim $H=2$. But $H\subset K\subset T$ with dim $T\leqq 1$, which is impossible [2, p. 26]. Thus the curve must be plane.

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UNITARY MULTIPLES OF A MATRIX

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Canonical forms for one-sided multiples of a matrix have been considered from time to time. Examples are Schmidt's Theorem (Turnbull and Aitken [7], p. 96), the Hermite form (MacDuffee [4], p. 35), and Brenner [1]. A slightly changed version of an old theorem is given here. Only $n\times n$ matrices over the complex field and vectors over the n-dimensional complex space are considered, although results similar to those in the first part of this paper could be obtained for $m\times n$ matrices.

Given any two non-zero vectors ξ, η, there exists a non-singular matrix P such that $P\xi=\eta$. To see this let Q be a non-singular matrix with ξ as its first column and R be a non-singular matrix with η as its first column. Then $(RQ^{-1})Q=R$, and so $(RQ^{-1})\xi=\eta$. If in addition ξ and η have the same norm, *i.e.*, $\sqrt{\xi^*\xi}=\sqrt{\eta^*\eta}=1/a$, then $a\xi$ and $a\eta$ are normalized and so we may choose unitary matrices R and Q such that $(RQ^{-1})a\xi=a\eta$. In particular for any vector ξ there exists a unitary matrix U such that $U\xi=[a, 0, \cdots, 0]^T$ where $\xi^*\xi=a^2$ and $a>0$ if $\xi\neq 0$.

An immediate consequence of this assertion is that the triangular form given in the following theorem may be obtained.

THEOREM 1. *For any matrix A there exists a unitary matrix U such that $UA=T=(c_{ij})$ has these properties:*

(i) $c_{ij}=0$ *for* $i>j$,

(ii) If c_{ik} is the first non-zero element in the i-th row then $c_{ik}>0$ and all other elements in the first k columns of the i-th through the n-th rows are zero.

The matrix UA is unique. If A is non-singular then U is also unique.

COROLLARY (Schmidt's Theorem). *If A is non-singular there exists a (unique) triangular matrix such that AT is unitary, where T has the form described above.*

To show uniqueness of the form given in the theorem suppose that $VA=(d_{ij})$ also has this form where V is unitary. If $c_{1k}>0$ is the first non-zero element in the first row of UA, then c_{1k} is the norm of the k-th column of A, since a unitary matrix does not change the norm of a vector, and so $c_{1k}=d_{1k}$ and d_{1k} is the first non-zero element in the first row of VA. Let $VU^*=(b_{ij})$. Since $VU^*(UA)=VA$ and $c_{1k}=d_{1k}\neq 0$, it follows that $b_{11}=1$. But as VU^* is unitary this implies $b_{1j}=b_{i1}=0$, for $i, j, \neq 1$. Thus the first rows of UA and VA are identical. Similarly, if c_{2k} is the first non-zero element of the second row of UA then d_{2k} is the first non-zero element of the second row of VA. Hence $d_{2k}>0$ and so $c_{2k}=d_{2k}\neq 0$. Again this implies $b_{22}=1$, $b_{2j}=b_{i2}=0$, for $i, j, \neq 2$, and thus that the second rows of UA and VA coincide. Continuation of the process shows that the first r rows of UA and VA coincide where r is the rank of A. Hence the form is unique as the last $n-r$ rows of both UA and VA are 0. If A is non-singular, $VU^*=I$ and $V=U$, although this is quite obvious anyway.

Here T has 0's below the main diagonal, but we could just as easily have placed them above. It is also clear that dual results hold for right multiples.

This, then, is the simplest form to which a matrix can be reduced by one-sided multiplication by a unitary matrix. The question naturally arises as to what additional simplification can be accomplished by a general non-singular matrix. Certainly any positive c_{ik} mentioned in (ii) above can be reduced to 1, and all other elements in the k-th column can be reduced to 0. This is essentially all that can be done. In fact, this is the familiar Hermite form, except for a permutation of the rows.

THEOREM 2. *For any matrix A there exists a non-singular matrix P such that $PA=H=(c_{ij})$ has these properties:*

(i) $c_{ij}=0$ *for* $i>j$,

(ii) *If c_{ik} is the first non-zero element in the i-th row then $c_{ik}=1$, all other elements in the first k columns of the i-th through the n-th rows are* 0, *and all other elements in the k-th column are* 0.

The matrix PA is unique. If A is non-singular then P is also unique.

COROLLARY. *The Hermite form of A is unique.*

To show uniqueness suppose that $P_1A=H_1$ and $P_2A=H_2$ where H_1 and H_2 have the form above. Let $P_1U_1=T_1$ and $P_2U_2=T_2$ be the right unitary canonical forms for P_1 and P_2 (with 0's below the main diagonal) under the dual result to Theorem 1, where U_1 and U_2 are unitary. Then $U_1^*A=T_1^{-1}H_1$ and $U_2^*A=T_2^{-2}H_2$ each have the form prescribed in Theorem 1 and so are equal, for as P_i is non-singular, T_i has positive elements down the main diagonal. Hence from $(T_1T_2^{-1})H_2=H_1$ we see that $T_1T_2^{-1}$ has 1's down the main diagonal in the first r rows, where r is the rank of A. Because of the form of H_i (1 being the first non-zero element in a row and all other elements in the column containing the 1 being 0) this implies $T_1T_2^{-1}$ has I_r, the $r\times r$ identity matrix, in the upper left hand corner. Since the last $n-r$ rows of both H_1 and H_2 are 0, we see that H_1 coincides with H_2.

Finally, two-sided unitary multiples of a commutative set of matrices will be considered.

THEOREM 3 (Rademacher [6]). *Any set of commutative matrices has a common eigenvector.*

Since there are only a finite number of linearly independent matrices, it suffices to consider finite sets. The proof is by induction on the number m of matrices in the set. It is certainly true for $m=1$. Suppose then it is true for sets with $m-1$ matrices and consider the set $\{A_1, \cdots, A_m\}$ of commutative matrices. By hypothesis $\{A_1, \cdots, A_{m-1}\}$ have a common eigenvector ξ. In the set $\xi, A_m\xi, A_m^2\xi, \cdots$, let $A_m^{k+1}\xi$ be the first vector linearly dependent on the preceding vectors. Then the space spanned by $\xi, A_m\xi, A_m^2\xi, \cdots, A_m^k\xi$ is invariant under A_m and hence contains an eigenvector η of A_m. But every vector in this space is an eigenvector of $A_1, \cdots, A_{m-1}$, since $A_j(A_m^r\xi)=A_m^r(A_j\xi)=\lambda(A_m^r\xi)$. Hence η is a common eigenvector of $A_1, \cdots, A_m$.

COROLLARY 1. *Any set of commutative matrices may be simultaneously unitarily triangularized.*

The proof by induction on the order n of the matrices is the same as that usually given for triangularization of a single matrix.

COROLLARY 2. (Frobënius). *If $A_1, \cdots, A_m$ are commutative there exists an ordering $\lambda_1^i, \cdots, \lambda_m^i$ of the roots of A_i such that for any rational function $f(x_1, \cdots, x_m)$ the characteristic roots of $f(A_1, \cdots, A_m)$ are $f(\lambda_j^1, \lambda_j^2, \cdots, \lambda_j^m)$, $j=1, 2, \cdots, n$.*

This is an immediate consequence of Corollary 1 and the nice behaviour of the diagonal elements of triangular matrices under addition and multiplication.

The ordering of the roots is determined by the common eigenvector and so is independent of the rational function.

This theorem and corollaries give a sufficient but not a necessary condition. It is much more difficult to give both necessary and sufficient conditions; see

McCoy [5]. A more elementary proof of McCoy's Theorem is given by Drazin, Dungey, and Gruenberg [2].

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APPROXIMATION FORMULAS FOR ELLIPTIC INTEGRALS

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1. Elliptic integrals of the second kind. k^2 near one and ϕ small. Consider the elliptic integral of the second kind

$$(1) \qquad E(\phi, k) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \phi}\, d\phi = \int_0^\phi \cos \phi \sqrt{1 + h \tan^2 \phi}\, d\phi$$

where $h = 1 - k^2$. Expanding the radical by the Binomial Theorem we get for the integrand

$$(2) \quad \cos \phi \left[1 + \frac{h}{2} \tan^2 \phi - \frac{h^2}{8} \tan^4 \phi + \frac{h^3}{16} \tan^6 \phi - \frac{5h^4}{128} \tan^8 \phi + \frac{7h^5}{256} \tan^{10} \phi - \cdots \right].$$

For $h \tan^2 \phi < 1$ this series converges absolutely and uniformly and may be integrated termwise, obtaining

$$(3) \qquad E(\phi, h) = A \sin \phi + B \log_e \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) + C \tan \phi \sec \phi + D \tan \phi \sec^3 \phi + E \tan \phi \sec^5 \phi \cdots$$

where

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$$A = 1 - \frac{h}{2} - \frac{h^2}{8} - \frac{h^3}{16} - \frac{5h^4}{128}$$

$$B = \frac{h}{2} + \frac{3h^2}{16} + \frac{15h^3}{128} + \frac{175h^4}{2048}$$

$$C = -\frac{h^2}{16} - \frac{9h^3}{128} - \frac{145h^4}{2048}$$

$$D = \frac{h^3}{64} + \frac{95h^4}{3072}$$

$$E = -\frac{5h^4}{768}.$$

Let E_i be the sum of the first i terms of (3). Let R_i be the numerical value of the remainder associated with E_i. That is, $|E(\phi, h) - E_2| = R_2$, *etc.* From (2) it is obvious that

$$R_2 < \frac{h^2}{8}\int_0^\phi \cos\phi \tan^4\phi\, d\phi = \frac{h^2}{8} I_2$$

$$R_3 < \frac{h^3}{16}\int_0^\phi \cos\phi \tan^6\phi\, d\phi = \frac{h^3}{16} I_3$$

$$R_4 < \frac{5h^4}{128}\int_0^\phi \cos\phi \tan^8\phi\, d\phi = \frac{5h^4}{128} I_4$$

$$R_5 < \frac{7h^5}{256}\int_0^\phi \cos\phi \tan^{10}\phi\, d\phi = \frac{7h^5}{256} I_5.$$

Evidently for k^2 near unity and ϕ small E_2, E_3, E_4, E_5 give increasingly accurate approximations to $E(\phi, k)$. Note in particular the simple and compact formula

$$E_2 = \left(1 - \frac{h}{2}\right)\sin\phi + \frac{h}{2}\log_e \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \tag{5}$$

which gives surprisingly good results for h and ϕ small.

To illustrate the degree of accuracy that might be expected of our formulas we shall approximate $E(\phi, k)$ for $\phi = 30°$ and $k^2 = .92$. Here $h = .08$ and we get

$$E_2 = .501972$$

$$E_3 = .50196475$$

$$E_4 = .5019648160$$

$$E_5 = .50196481508.$$

The last result is in error by about one in the last decimal.

Table I below is useful in estimating the error involved in using E_2, E_3, E_4, and E_5.

TABLE I

ϕ	I_2	I_3	I_4	I_5
18°	.00057	.000046	.000004	.000000
24°	.00279	.000398	.000061	.000011
30°	.00929	.002171	.000555	.000150
36°	.02532	.009199	.003690	.001572
42°	.06111	.037872	.020422	.013273

This table tells us that for $\phi = 30°$, say, and any acceptable value of h, the error involved in using E_5 as an approximation to $E(\phi, k)$ is measured by

$$R_5 < \frac{7h^5}{256} I_5 = \frac{7}{256}(.000150)h^5 = .000004h^5.$$

Thus in our problem E_5 gives a result in error by less than $(.000004)(.08)^5 = .000\,000\,000\,02$. In fact, $|E_5 - E(\phi, k)| < .000012$ over the wide range of values: $0 \leqq \phi \leqq 42°$, $.5 \leqq k^2 \leqq 1.0$.

Form (3) is easy to evaluate since very accurate tables of the trigonometric functions are quite common and extensive tables of log tan $(\frac{1}{4}\pi + \frac{1}{2}\phi)$ have been constructed by Legendre and Gudermann and are available, though not exactly common.

Another useful approximation to $E(\phi, k)$ may be obtained by expanding the terms in (2) into a Taylor's series and integrating termwise obtaining

$$E(\phi, h) = \phi + A\phi^3 + B\phi^5 + C\phi^7 + D\phi^9 + E\phi^{11} + \cdots \tag{6}$$

where the coefficients are defined by

$$A = -\frac{1}{6} + \frac{h}{6}$$

$$B = \frac{1}{120} + \frac{h}{60} - \frac{h^2}{40}$$

$$C = -\frac{1}{5040} + \frac{31h}{5040} - \frac{5h^2}{336} + \frac{h^3}{112}$$

$$D = \frac{1}{362880} + \frac{173h}{90720} - \frac{23h^2}{2880} + \frac{h^3}{96} - \frac{5h^4}{1152}$$

$$E = \frac{-1}{39916800} + \frac{25261h}{39916800} - \frac{1063h^2}{266112} + \frac{181h^3}{21120} - \frac{65h^4}{8448} + \frac{7h^5}{2816}.$$

The above form is well adapted to a CPC-IBM machine since the coefficients can be evaluated and the series summed in one operation. Forms (3) and (6)

both give quite accurate approximations to $E(\phi, k)$, for small ϕ and k^2 near one, *i.e.*, h small.

2. Elliptic integrals of the first kind. k^2 near one and ϕ small.

Elliptic integrals of the first kind may be treated in a similar manner. Thus

$$F(\phi, k) = \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \int_0^\phi \sec \phi (1 + h \tan^2 \phi)^{-1/2} d\phi, \tag{7}$$

where again, $h = 1 - k^2$.

Expanding the integrand by the Binomial Theorem we get the convergent power series

$$\sec \phi \left[1 - \frac{h}{2} \tan^2 \phi + \frac{3h^2}{8} \tan^4 \phi - \frac{5h^3}{16} \tan^6 \phi + \frac{35h^4}{128} \tan^8 \phi - \frac{63h^5}{256} \tan^{10} \phi + \cdots \right] \tag{8}$$

which may be integrated termwise in the region of convergence giving

$$F(\phi, h) = A \log_e \tan \left(\frac{\pi}{4} + \frac{\phi}{2} \right) + B \tan \phi \sec \phi + C \tan \phi \sec^3 \phi + D \tan \phi \sec^5 \phi + E \tan \phi \sec^7 \phi + \cdots \tag{9}$$

where

$$A = 1 + h/4 + 9h^2/64 + 25h^3/256 + 1225h^4/16384$$
$$B = -h/4 - 15h^2/64 - 55h^3/256 - 3255h^4/16384$$
$$C = 3h^2/32 + 65h^3/384 + 5705h^4/24576$$
$$D = -5h^3/96 - 875h^4/6144$$
$$E = 35h^4/1024.$$

Letting F_i represent the sum of the first i terms of (9) and R_i^* the associated remainders we have

$$R_2^* < \frac{3h^2}{8} \int_0^\phi \sec \phi \tan^4 \phi d\phi = \frac{3h^2}{8} I_2^*$$

. . . . . .

$$R_5^* < \frac{63h^5}{256} \int_0^\phi \sec \phi \tan^{10} \phi d\phi = \frac{63h^5}{256} I_5^*.$$

A fairly accurate approximation to $F(\phi, h)$ is given by the abbreviated formula

$$F_2 = \left(1 + \frac{h}{4}\right) \log_e \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) - \frac{h}{4} \tan\phi \sec\phi. \tag{10}$$

If the F_i are evaluated for $\phi = 30°$, $k^2 = .92$ we get

$$F_2 = .54696$$
$$F_3 = .5469867$$
$$F_4 = .54698621$$
$$F_5 = .5469862140.$$

The table below is useful in getting a bound for the error involved in using F_i as an approximation to $F(\phi, k)$.

TABLE II

ϕ	I_2^*	I_3^*	I_4^*	I_5^*
18°	.00061	.00005	.000004	.000000
24°	.00319	.00046	.000071	.000012
30°	.01146	.00273	.000706	.000193
36°	.03452	.01289	.005263	.002270
42°	.09460	.05391	.033695	.022273

Thus for $\phi = 30°$, say, we have

$$R_2^* < \frac{3h^2}{8}(.01146) = (.0043)h^2$$

and

$$R_5^* < \frac{63h^5}{256} I_5^* = (.000046)h^5.$$

For the relative wide range ($0 \leqq \phi \leqq 36°$, $.7 \leqq k^2 \leqq 1.0$) the table shows that $|F(\phi, k) - F_5| < .000015$.

If (8) is expanded into a Taylor's series and integrated termwise we get

$$F(\phi, h) = \phi + A\phi^3 + B\phi^5 + C\phi^7 + D\phi^9 + \cdots \tag{11}$$

where

$$A = (1/2 - h/2)/3$$
$$B = (5/24 - 7h/12 + 3h^2/8)/5$$
$$C = (61/720 - 331h/720 + 11h^2/16 - 5h^3/16)/7$$
$$D = (277/8064 - 3071h/10080 + 249h^2/320 - 25h^3/32 + 35h^4/128)/9.$$

Both (9) and (11) give good results when h and ϕ are small.

3. Elliptic integrals of the second kind. k^2 small.

If k^2 is small (say, $k^2<.5$) and ϕ large (say, $\phi>\pi/4$) the above approximations may be unsatisfactory. In such a case we transform (1) as follows:

$$
\begin{aligned}
E(\phi, k) &= \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \phi}\, d\phi = \int_0^{\phi} \sqrt{1 - \frac{k^2(1 - \cos 2\phi)}{2}}\, d\phi \\
&= \frac{1}{\sqrt{1+M}} \int_0^{\phi} \sqrt{1 + M \cos 2\phi}\, d\phi \\
&= \frac{1}{2\sqrt{1+M}} \int_0^{x-2\phi} \sqrt{1 + M \cos x}\, dx,
\end{aligned} \tag{12}
$$

where $M=k^2/(2-k^2)$.

Expanding the radical by the Binomial Theorem and integrating termwise we get

$$
\begin{aligned}
E(x, M) = \frac{1}{2\sqrt{1+M}} \Big[& I_0 + \frac{M}{2} I_1 - \frac{M^2}{2^3} I_2 + \frac{M^3}{2^4} I_3 - \frac{5M^4}{2^7} I_4 + \frac{7M^5}{2^8} I_5 \\
& - \frac{21M^6}{2^{10}} I_6 + \frac{33M^7}{2^{11}} I_7 - \frac{429M^8}{2^{15}} I_8 + \frac{715M^9}{2^{16}} I_9 - \frac{2431M^{10}}{2^{18}} I_{10} \\
& + \frac{4199M^{11}}{2^{19}} I_{11} - \frac{29393M^{12}}{2^{22}} I_{12} + \frac{52003M^{13}}{2^{23}} I_{13} \\
& - \frac{185725M^{14}}{2^{25}} I_{14} + \cdots \Big]
\end{aligned} \tag{13}
$$

where

$$
\begin{aligned}
I_0 &= \int_0^x dx, \qquad I_1 = \int_0^x \cos x dx = \sin x \\
I_2 &= \int_0^x \cos^2 x dx = \frac{\sin x \cos x}{2} + \frac{x}{2}, \\
I_3 &= \int_0^x \cos^3 x dx = \frac{\sin x \cos^2 x}{3} + \frac{2}{3} I_1, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
I_n &= \int_0^x \cos^n x dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}.
\end{aligned}
$$

Since I_n approaches zero monotonically, it is obvious that (13) converges rapidly for $M \ll 1$. The integrals are easy to evaluate since they depend on a recursion formula.

In the case of the complete elliptic integral we have

$$E(M) = \frac{\pi}{2\sqrt{1+M}}\left[1 - .0625M^2 - .0146484375M^4 - .0064086914M^6 - .0035798550M^8 - .0022821575M^{10} - .0015808695M^{12} - .0011594388M^{14} - \cdots\right]. \tag{14}$$

Series (14) converges rapidly for relatively large values of k^2. Thus for $k^2=.5$ $(M=1/3)$ we get $E_2=1.3509$, $E_3=1.35066$, $\cdots$ $E_7=1.3506438818$, a result in error by possibly one in the last decimal place. For $k^2=.75$ $(M=.6)$ the series begins to converge rather slowly. It gives $E_8=1.2110564$, a result in error by .0000004.

4. Elliptic integrals of the first kind. k^2 small.

In this case

$$F(\phi, k) = \int_0^{\phi} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\sqrt{1+M}}{2}\int_0^{x=2\phi} \frac{dx}{\sqrt{1+M\cos x}}, \tag{15}$$

where again $M=k^2/(2-k^2)$.

Proceeding as before we get

$$F(x, M) = \frac{\sqrt{1+M}}{2}\left[I_0 - \frac{M}{2}I_1 + \frac{3M^2}{2^3}I_2 - \frac{5M^3}{2^4}I_3 + \frac{35M^4}{2^7}I_4 - \frac{63M^5}{2^8}I_5 + \frac{231M^6}{2^{10}}I_6 - \frac{429M^7}{2^{11}}I_7 + \frac{6435M^8}{2^{15}}I_8 - \frac{12155M^9}{2^{16}}I_9 + \frac{46189M^{10}}{2^{18}}I_{10} - \frac{88179M^{11}}{2^{19}}I_{11} + \frac{676039M^{12}}{2^{22}}I_{12} - \cdots\right] \tag{16}$$

where the I_n are exactly those given in series (13). For the complete integral we have

$$F(M) = \frac{\pi\sqrt{1+M}}{2}\left[1 + .1875M^2 + .1025390625M^4 + .070495605M^6 + .0536978M^8 + .043361M^{10} + .03636M^{12} + \cdots\right]. \tag{17}$$

Formulas (14) and (16) converge far more rapidly than do formulas (524) and (525) in *A Short Table of Integrals* by Peirce.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville, Tenn.

AN INEQUALITY FOR CONVEX FUNCTIONS

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In what follows, m and n are positive integers, x a non-negative real number and $f(x)$ a real function of x. We call the set of real numbers $a_1, \cdots, a_m$ an m-set if

$$a_1 \geqq a_2 \geqq a_3 \geqq \cdots \geqq a_m \geqq 0.$$

All our subsequent statements are true if no further restrictions are imposed on x and a's or if, on the other hand, for some fixed $b>0$, we insist that $x \leqq b$ and $a_1 \leqq b$; that is, we may work indifferently in a fixed finite interval or in the semi-infinite interval $x \geqq 0$.

We use $P(m)=P(m, f)$ to denote the proposition (true for some f, false for others):

For every m-set

$$f(a_1) - f(a_2) + \cdots + f(a_m) \geqq f(a_1 - a_2 + \cdots + a_m) \ (m \text{ odd}),$$

$$f(a_1) - f(a_2) + \cdots - f(a_m) \geqq f(a_1 - a_2 + \cdots - a_m) - f(0) \ (m \text{ even}).$$

Payne and Weinstein conjectured and Weinberger [*Proc. Nat. Acad. Sci.*, vol. 38, 1952, pp. 611–613] recently proved the special case of $P(m, f)$ in which $f(x)=x^r$ and $r \geqq 1$. In an abstract of Weinberger's paper [*Math. Reviews*, vol. 14, 1953, p. 24], Bellman supplied a simple proof that $P(m, f)$ (in a slightly weakened form) is true for any continuously differentiable convex f. Both Weinberger's and Bellman's proofs depend on differentiation. In fact, $P(m, f)$ for any continuous convex function f is a special case of Theorem 108 of Hardy, Littlewood and Pólya's *Inequalities* (Cambridge, 1934, hereafter referred to as HLP).

My object here is to show that the $P(m, f)$ for any given f have a very simple logical relationship among themselves, independent of any ideas of differentiation or of continuity. We use capital letters to denote propositions. $A+B \rightarrow C$ means "if A and B are both true, then C is true," and $A \equiv B$ means "$A \rightarrow B$ and $B \rightarrow A$."

All our statements refer to a fixed function f, which need not be continuous. We observe that nothing is changed in $P(m, f)$ if we replace every f by $f-c$ for some c independent of x. Hence, without loss of generality, we may suppose $f(0)=0$. For such an f, $P(m, f)$ reads

For every m-set,

$$\sum_{k=1}^{m}(-1)^{k-1}f(a_k) \geqq f\left(\sum_{k=1}^{m}(-1)^{k-1}a_k\right).$$

$P(1, f)$ is trivially true. If we put $a_{m+1}=0$, we see that

$$P(m+1, f) \to P(m, f). \tag{1}$$

Next, if we assume $P(m, f)$ and $P(3, f)$ and use them in succession, we have

$$\sum_{k=1}^{m+2}(-1)^{k-1}f(a_k) \geqq f(a_1) - f(a_2) + f\left(\sum_{k=3}^{m+2}(-1)^{k-1}a_k\right)$$

$$\geqq f\left(\sum_{k=1}^{m+2}(-1)^{k-1}a_k\right),$$

and this is $P(m+2, f)$. Hence

$$P(3, f) + P(m, f) \to P(m+2, f). \tag{2}$$

Using (2) to establish an obvious induction we have $P(3, f)\to P(2n+1, f)$ and $P(3, f)+P(2, f)\to P(2n, f)$, for every positive integer n. By (1), however, we see that $P(3, f)\to P(2, f)$ and so

$$P(3, f) \to P(m, f), \qquad (m \geqq 1). \tag{3}$$

By repeated use of (1), we see that

$$P(m, f) \to P(3, f), \qquad (m \geqq 3) \tag{4}$$

and so

$$P(m, f) \equiv P(3, f), \qquad (m \geqq 3). \tag{5}$$

Finally, while $P(3, f)\to P(2, f)$, the converse is false. For example, let $f(x)$ be defined as

$$0\ (0 \leqq x < 1), \qquad x-1\ (1 \leqq x < 2), \qquad 1\ (2 \leqq x < 3), \qquad x-2\ (x \geqq 3).$$

Then $P(2, f)$ is true, as may be easily seen from a figure. But $P(3, f)$ is false; for example,

$$f(3) - f(2) + f(1) = 1 - 1 + 0 = 0, \qquad f(3-2+1) = f(2) = 1.$$

Hence $P(3, f)$ is the fundamental inequality. If we put $a_1=x_1+\delta$, $a_2=x_1$, $a_3=x_2$, we see that $P(3, f)$ is equivalent to

$$f(x_1+\delta) - f(x_1) \geqq f(x_2+\delta) - f(x_2) \tag{6}$$

for all $\delta>0$ and all $x_1\geqq x_2\geqq 0$. This may be taken as a definition of convexity, in which case we have shown that, for any $m\geqq 3$, the truth of $P(m, f)$ is a necessary and sufficient condition for f to be convex.

HLP take

$$f(x_3) + f(x_4) \geqq 2f\left(\frac{x_3 + x_4}{2}\right) \tag{7}$$

for all $x_3 \geqq 0$, $x_4 \geqq 0$ as the definition of convexity. Relation (6) implies (7) and, for continuous $f(x)$, (7) implies (6) (HLP Theorem 86). If there is any $f(x)$ which satisfies (7) and not (6), it must be discontinuous and so (HLP Theorem 111) unbounded in any finite interval. The only known examples of discontinuous convex functions depend for their construction on Zermelo's Axiom of Choice (HLP p. 96). These examples satisfy both (6) and (7), so that whether functions satisfying (7) and not (6) exist is unknown, even if Zermelo's axiom is true.

A NOTE ON COMPLETE RESIDUE SYSTEMS

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The following is known [1]:

THEOREM. *If m is an integer $\geqq 3$, and if $\{a_i\}$, $\{b_i\}$, $i = 1, \cdots, m$, are two complete residue systems (mod m), then $\{a_i b_i\}$ is not a complete residue system (mod m).*

We here offer a simpler proof.

The theorem is well-known [2] for $m = p$ an odd prime. Indeed, excluding the zero element, by Wilson's Theorem the product of the remaining elements of a complete residue system is congruent to -1 (mod p); since this is true for $\{a_i\}$, $\{b_i\}$, it must fail for $\{a_i b_i\}$.

Evidently the theorem holds for $m = 4$. We proceed by multiplicative induction. We assume the theorem for m arbitrary, $m \geqq 3$. Let p be an arbitrary prime; suppose that $\{a_i b_i\}$, $i = 1, \cdots, mp$, is a complete residue system (mod mp). Now, every complete residue system (mod mp) contains exactly m multiples of p. Hence $p \mid a_i$ if and only if $p \mid b_i$, else $\{a_i b_i\}$ will contain more than m multiples of p. We may suppose $a_j = a_j' p$, $b_j = b_j' p$, $j = 1, \cdots, m$, where $\{a_j'\}$, $\{b_j'\}$ form complete residue systems (mod m). Thus the set $\{a_i b_i\}$ contains the set $\{a_j' b_j' p^2\}$. The incongruence of the elements $a_j' b_j' p^2$ (mod mp) implies the incongruence of the elements $a_j' b_j' p$ (mod m), which implies the incongruence of the elements $a_j' b_j'$ (mod m). This is contrary to our hypothesis on m; hence the supposition that $\{a_i b_i\}$ is a complete residue system (mod mp) is false, and the theorem is established.

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SOME TRIGONOMETRIC, HYPERBOLIC AND ELLIPTIC APPROXIMATIONS

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In this paper we shall present some close inequalities involving trigonometric and hyperbolic functions, and some estimates for the incomplete elliptic functions of first and second kind.

When the prismoid formula is used to evaluate the average value of $f(x)=\cos x$ on the interval $-\theta\leqq x\leqq\theta$ we obtain

$$\frac{\sin\theta}{\theta}=\frac{1}{2\theta}\int_{-\theta}^{\theta}\cos x\,dx\sim\frac{\cos\theta+4+\cos\theta}{6}=\frac{2+\cos\theta}{3}. \tag{1}$$

This approximation is equivalent to approximating a small angle θ in a right triangle by $3\sin\theta/(2+\cos\theta)$, and thus leads to the formula [4]

$$A^\circ=\frac{a}{2c+b}\,172^\circ \tag{2}$$

for the smallest angle A° in a right triangle with sides $a<b<c$. W. A. Hurwitz [7] pointed out that this remarkable formula (2) was published in the works of Nicolaus Cusanus [3] in 1514, and in 1699 by Ozanam [8]. Many approximations for arcs and chords of circles, such as those given by J. M. Bruce [1, 2], are based either directly on (1) or are equivalent to it.

The accuracy of (1) can be considerably improved by taking account of its error of about $\theta^4/180$ in the following way. We write

$$1-\frac{\sin\theta}{\theta}=\frac{1-\cos\theta}{3-q\theta^2/10} \tag{3}$$

where q is a function of θ. We find that q varies monotonically from 1 to $10/\pi^2=1.013212$ as θ varies from 0 to π, and thus obtain the strong inequalities

$$\frac{\sin\theta(3-\theta^2/10)}{2+\cos\theta-\theta^2/10}<\theta<\frac{\sin\theta(3-\theta^2/\pi^2)}{2+\cos\theta-\theta^2/\pi^2},\qquad\text{for } 0<\theta<\pi. \tag{4}$$

For $\theta=\pi/2$ and $\theta=\pi/3$ these inequalities for $\theta/\sin\theta$ become

$$\frac{3-\pi^2/40}{2-\pi^2/40}=\frac{11.1304}{7.1304}<\frac{\pi}{2}<\frac{11}{7}, \tag{4a}$$

$$\frac{3-\pi^2/90}{2.5-\pi^2/90}=\frac{52.2608}{43.2608}<\frac{2\pi}{3\sqrt{3}}<\frac{52}{43}. \tag{4b}$$

Actually the approximation 78/43 for $\pi/\sqrt{3}$ is the best rational approximation for this number with denominator less than 100, and its relative error is less than 0.0001.

To establish the fact that the function $q(\theta)$ in (3) increases monotonically as θ varies from 0 to π, and then decreases as θ varies from π to 2π, we may use the

following device. We first differentiate the expression

$$\frac{q}{10} = \frac{3}{\theta^2} - \frac{1-\cos\theta}{\theta(\theta-\sin\theta)} \tag{5}$$

and then multiply by the factor $\theta^2(\theta-\sin\theta)^2/\sin\theta$ in order to simplify the denominator and remove the root of $dq/d\theta$ at $\theta=\pi$. We then expand the resulting function $\phi(\theta)$ as an infinite sum of functions that are each positive on the interval $0<\theta<2\pi$, as follows:

$$\begin{aligned}\phi(\theta) &= \frac{\theta^2(\theta-\sin\theta)^2}{10\sin\theta}\,\frac{dq}{d\theta} \\ &= (\cos\theta - 1 + \theta^2/2) + 6\{1-(\sin\theta)/\theta\} \\ &\quad + 6\{1-(\theta/2)\cot(\theta/2)\} - 3\theta^2/2 \\ &= (48-\theta^2)\theta^8/10! + \sum_{n=3}^{\infty} p_n(\theta),\end{aligned} \tag{6}$$

where the even polynomials $p_n(\theta)$ defined by

$$\begin{aligned}p_n(\theta) = \frac{\theta^{4n}}{(4n)!} - \frac{\theta^{4n+2}}{(4n+2)!} + \frac{6\theta^{4n-2}}{(4n-1)!} - \frac{6\theta^{4n}}{(4n+1)!} \\ + 6B_{2n-1}\frac{\theta^{4n-2}}{(4n-2)!} + 6B_{2n}\frac{\theta^{4n}}{(4n)!}\end{aligned} \tag{7}$$

are positive on the interval $0<\theta<2\pi$, for $n>1$. The positive coefficients B_n that appear in the expansion of $1-(\theta/2)\cot(\theta/2)$ are the Bernoulli numbers 1/6, 1/30, 1/42, 1/30, 5/66, *etc.* Since $\phi(\theta)$ is positive, it follows that $dq/d\theta$ has the same sign as $\sin\theta$ on the interval $0<\theta<2\pi$.

In the second inequality of (4) we set $\theta=\pi x$ and examine the error by studying the function $\epsilon(x)$:

$$\epsilon(x) = \frac{\sin\pi x}{\pi x} + \frac{1-\cos\pi x}{3-x^2} - 1 \sim \frac{(x^2-x^4)^2}{143-20x^2}, \qquad 0<x^2<1. \tag{8}$$

The right member of (8), like $\epsilon(x)$, vanishes to the fourth order for $x=0$, and to the second order for $x=\pm 1$, and when the ratio $(x^2-x^4)^2/\epsilon(x)$ is plotted against $u=x^2$, it is closely fitted by the line $143-20u$ for values of u between 0 and 1. Thus the maximum value of $\epsilon(x)$ on $[0, 1]$ is only about 0.0004.

Similar inequalities involving hyperbolic functions [6] arise when we change the sign of θ^2 in (3). We have

$$\frac{\sinh u}{u} - 1 = \frac{\cosh u - 1}{3 + qu^2/10}, \qquad 0<q\leqq 1, \tag{9}$$

where $q\to 1$ when $u\to 0$; but here q decreases from 1 to about 0.91 as u increases

from 0 to 5. Hence

$$\frac{\sinh u(3+u^2/11)}{2+\cosh u+u^2/11} < u < \frac{\sinh u(3+u^2/10)}{2+\cosh u+u^2/10}, \quad \text{for } 0<u<5. \tag{10}$$

The substitution $u=\ln z$ gives inequalities for estimating logarithms:

$$\frac{(z^2-1)(3+u^2/11)}{z^2+4z+1+2zu^2/11} < \ln z < \frac{(z^2-1)(3+u^2/10)}{z^2+4z+1+2zu^2/10}, \qquad u=\ln z<5. \tag{11}$$

Thus for example, for $z=2$ we have

$$\frac{9+3u^2/11}{13+4u^2/11} < \ln 2 < \frac{9+3u^2/10}{13+4u^2/10}, \quad u=\ln 2=.693147, \tag{11a}$$

and the approximation $9/13=0.6923$ is too small. But if we substitute $u=9/13$ in the right member of (11a) we obtain the value 0.693146, with error about 10^{-6}.

We now attempt to extend an approximation like (3) to elliptic functions F and E of the first and second kinds:

$$F=\int_0^\theta \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \qquad E=\int_0^\theta \sqrt{1-k^2\sin^2\phi}\,d\phi. \tag{12}$$

Clearly for $k^2=0$, both F and E reduce to θ. When the elliptic functions $\sin\theta=\operatorname{sn} F$, $\cos\theta=\operatorname{cn} F$, $(1-k^2\sin^2\theta)^{1/2}=\operatorname{dn} F$ are expanded as power series in F we observe that in the series for $(\operatorname{sn} F)/F$ the coefficients of F^{2n} are reciprocal polynomials of degree n in k^2, so that $(\operatorname{sn} F)/F$ is unchanged by the substitution $(k, F)\to(1/k, kF)$. However, this substitution interchanges the functions $\operatorname{cn} F$ and $\operatorname{dn} F$. Hence we try a modification of (3) in the form

$$1-\frac{\operatorname{sn} F}{F}\sim\frac{1-\operatorname{cn} F}{3-aF^2/10}+\frac{1-\operatorname{dn} F}{3-bF^2/10} \tag{13}$$

and find that

$$a+k^2b=1-k^2+k^4, \qquad b(k)=k^2a(1/k). \tag{14}$$

The necessary conditions (14) are met by taking $a=1-k^2/2$, $b=k^2-1/2$, or $a=b=(1-k^2+k^4)/(1+k^2)$. We find that the function

$$\frac{\operatorname{sn} F}{F}+\frac{1-\operatorname{cn} F}{3-F^2(2-k^2)/20}+\frac{1-\operatorname{dn} F}{3-F^2(2k^2-1)/20}-1 \tag{15}$$

behaves like $\theta^6(-1+12k^2+12k^4-k^6)/15(7!)$ for θ near 0, and is near zero for all values of F between 0 and π. (It is not near 0 for large F, however.)

Nearly as good is the approximation obtained from (13) taking $a=b$, which gives a formula similar to (4)

$$F = \int_0^\theta (1 - k^2 \sin^2 \phi)^{-1/2} d\phi = \frac{\sin \theta(3 - a\theta^2/10)}{1 + (1 - k^2 \sin^2 \theta)^{1/2} + \cos \theta - a\theta^2/10},$$

(16)

$$a \sim \frac{1 - k^2 + k^4}{1 + k^2}.$$

A similar result obtained for elliptic integrals of the second kind is

$$E = \int_0^\theta (1 - k^2 \sin^2 \phi)^{1/2} d\phi = \frac{\sin \theta(3 - q\theta^2/10)}{3 - (1 - k^2 \sin^2 \theta)^{1/2} + \cos \theta - q\theta^2/10}, \tag{17}$$

$$q \sim 2 + k^2 - (1 - k^2 \sin^2 \theta).$$

Formula (17) reduces to an identity for $k^2 = 1$, whereas formula (16) may be refined to the inequality

$$\frac{\sin \theta(3 - \theta^2/20)}{1 + 2 \cos \theta - \theta^2/20} < \int_0^\theta \sec \phi d\phi = \ln (\sec \theta + \tan \theta) \tag{18}$$

$$< \frac{\sin \theta(3 - \theta^2/10)}{1 + 2 \cos \theta - \theta^2/10}, \qquad 0 < \theta < \frac{5\pi}{12}.$$

Thus the meridian distance on a Mercator chart from the equator to a latitude $\theta < 57°$ may be approximated by $3 \sin \theta/(1+2 \cos \theta)$ with about the same relative error ($<1\%$) that θ is approximated by $3 \sin \theta/(2+\cos \theta)$.

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AN INTEGRAL INEQUALITY

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1. In a recent note in this Monthly [4] one of the authors gave necessary and sufficient conditions for a continuous function $\Phi(t, u_1, \cdots, u_n)$, $0 \leqq t \leqq 1$, $u_i \geqq 0$, in order that

$$\int_0^1 \Phi(t, f_1, \cdots, f_n) dt \leqq \int_0^1 \Phi(t, f_1^*, \cdots, f_n^*) dt \tag{1}$$

for each set of positive bounded measurable functions $f_i(t)$, namely (5) and (6) below. By $f_i^*(t)$ we denote the decreasing rearrangement of $f_i(t)$. Two particular cases of (1) were given before by Ruderman [5]. In this note we discuss similar inequalities which govern the relation $f \prec g$. Theorem 1 gives necessary and sufficient conditions for a continuous function $\Phi(t, u_1, \cdots, u_n)$, defined for $0 \leqq t \leqq 1$ and all values of the u_i, in order that

$$\int_0^1 \Phi(t, f_1, \cdots, f_n)dt \leqq \int_0^1 \Phi(t, g_1, \cdots, g_n)dt \tag{2}$$

for each system of decreasing bounded functions f_i, g_i where $f_i \prec g_i$, $i=1, \cdots, n$. The relation $f \prec g$ (compare [3, p. 45]) means that

$$\int_0^x f(t)dt \leqq \int_0^x g(t)dt, \qquad 0 \leqq x \leqq 1, \tag{3}$$

and

$$\int_0^1 f(t)dt = \int_0^1 g(t)dt. \tag{4}$$

(Sometimes condition (4) is omitted in this definition, see §3.) If f_i is decreasing, then $f_i = f_i^*$, so that (2) is *not* a generalization of (1). Nevertheless the conditions on Φ in (2) are exactly the same as in (1), except that the u_i now may be negative and Φ must be convex in each of the u_i. If Φ satisfies all these conditions, we have as a combination of (1) and Theorem 1, that relation (2) holds if $f_i^* \prec g_i$, $i=1, \cdots, n$ and the g_i are decreasing. We indicate several special cases of (2) in §3.

2. We shall often omit those arguments of Φ which are the same in a certain formula, and, for a given i, $1 \leqq i \leqq n$, shall use the notation U_I for the set of all u_j with $j \neq i$. In [4] it has been shown that necessary and sufficient conditions for Φ in (1) are

$$\Phi(u_i + h, u_j + h) - \Phi(u_i + h, u_j) - \Phi(u_i, u_j + h) + \Phi(u_i, u_j) \geqq 0, \tag{5}$$

$$\int_0^\delta \{\Phi(a + \delta + t, u_i) - \Phi(a + \delta + t, u_i + h) + \Phi(a + t, u_i + h) - \Phi(a + t, u_i)\}dt \geqq 0, \tag{6}$$

for all $0 \leqq a \leqq 1 - 2\delta$, $\delta > 0$, $u_k \geqq 0$, $k=1, \cdots, n$, $h \geqq 0$ and $i \neq j$. We shall also use the condition of convexity

$$\Phi(u_i + h) - 2\Phi(u_i) + \Phi(u_i - h) \geqq 0 \tag{7}$$

for $i=1, \cdots, n$.

THEOREM 1. *Conditions* (5), (6) *and* (7) (*with the* u_k *now of arbitrary sign*) *are necessary and sufficient in order that* Φ *satisfies* (2).

PROOF. (a) *Necessity.* Let $0 \leqq a < a+2\delta \leqq 1$, $i \neq j$, $h_i \geqq 0$, $h_j \geqq 0$, u_k, $k=1, \cdots, n$ arbitrary. We put

$$f_i(t) = \begin{cases} u_i + h_i, & 0 \leqq t \leqq a \\ u_i, & a < t \leqq a + 2\delta, \\ u_i - h_i, & a + 2\delta < t \leqq 1, \end{cases} \qquad g_i(t) = \begin{cases} u_i + h_i, & 0 \leqq t \leqq a + \delta \\ u_i - h_i, & a + \delta < t \leqq 1, \end{cases}$$

$$f_j(t) = g_j(t) = \begin{cases} u_j + h_j, & 0 \leqq t \leqq a + \delta \\ u_j, & a + \delta < t \leqq 1, \end{cases}$$

$$f_k(t) = g_k(t) = u_k, \quad 0 \leqq t \leqq 1, \quad k \neq i, \quad k \neq j.$$

Then $f_k \prec g_k$ for $k=1, \cdots, n$. The inequality (2) reduces in this case to

$$\int_0^\delta \{\Phi(a + t, u_i + h_i, u_j + h_j) - \Phi(a + t, u_i, u_j + h_j) + \Phi(a + \delta + t, u_i - h_i, u_j) - \Phi(a + \delta + t, u_i, u_j)\} dt \geqq 0. \tag{8}$$

Putting here $h_j = 0$, dividing by δ and making $\delta \to 0$, we obtain (7).

To deduce (6), we recall that a convex continuous function is an integral of an increasing function. Let

$$\Phi(t, u_1, \cdots, u_n) = \int_0^{u_i} \phi(t, v, U_J) dv. \tag{9}$$

Inequality (8) for $h_i = h$, $h_j = 0$ may be written in the form

$$\int_0^\delta \left\{ \int_{u_i}^{u_i+h} \phi(a + t, v) dv - \int_{u_i-h}^{u_i} \phi(a + \delta + t, v) dv \right\} dt \geqq 0,$$

or also

$$\frac{1}{h} \int_{u_i}^{u_i+h} dv \int_0^\delta \phi(a + t, v) dt - \frac{1}{h} \int_{u_i-h}^{u_i} dv \int_0^\delta \phi(a + \delta + t, v) dt \geqq 0. \tag{10}$$

For fixed a and δ, both inner integrals are increasing functions of v. For all those v for which they both are continuous, hence a.e., we get from (10) for $h \to 0$:

$$\int_0^\delta \phi(a + t, v) dt - \int_0^\delta \phi(a + \delta + t, v) dt \geqq 0.$$

Integrating this with respect to v from u_i to $u_i + h$ and using the relation (9), we obtain (6).

Finally, dividing (8) by δ and making $\delta \to 0$, we get

$$\Phi(u_i + h_i, u_j + h_j) - \Phi(u_i, u_j + h_j) + \Phi(u_i - h_i, u_j) - \Phi(u_i, u_j) \geqq 0,$$

or, using (9),

$$\frac{1}{h_i}\int_{u_i}^{u_i+h_i}\phi(v, u_j + h_j)dv - \frac{1}{h_i}\int_{u_i-h_i}^{u_i}\phi(v, u_j)dv \geqq 0.$$

For $h_i \to 0$ we obtain that $\phi(v, u_j+h_j)-\phi(v, u_j)\geqq 0$ a.e. Integrating this from u_i to u_i+h_i with respect to v we obtain

$$\Phi(u_i + h_i, u_j + h_j) - \Phi(u_i, u_j + h_j) - \Phi(u_i + h_i, u_j) + \Phi(u_i, u_j) \geqq 0, \tag{11}$$

which reduces to (5) for $h_i=h_j=h$.

(b) *Sufficiency.* Clearly it is sufficient to prove (2) in the case when $f_k=g_k$ except for $k=i$, for the general result follows by successive application of the special one. We may further assume that all f_k and g_k are step-functions, constant on each of the intervals $(s/p, (s+1)/p)$, $s=0, \cdots, p-1$, for arbitrary f_k, g_k may be approximated by step-functions of this kind. In this case, by [3, p. 47, Lemma 2], f_i may be transformed into g_i by successive application of a finite number of transformations T. A transformation T changes the value u_i of f_i on an interval $(q/p, (q+1)/p)$ into u_i+h_i, the value u_i' of f_i on $(r/p, (r+1)/p)$ into $u_i'-h_i$, where $u_i' \leqq u_i$, $h_i \geqq 0$, $q<r$, and does not change the values of f_i on other intervals.† The increase of the integral

$$\int_0^1 \Phi(t, f_1, f_2, \cdots, f_n)dt$$

corresponding to a transformation T is equal to

$$\Delta = \int_0^\delta \{\Phi(a + l\delta + t, u_i' - h_i, U_J) - \Phi(a + l\delta + t, u_i', U_J) + \Phi(a + t, u_i + h_i, U_J + H_J) - \Phi(a + t, u_i, U_J + H_J)\}dt,$$

where $\delta=1/p$, $a=q/p$, $l=r-q$ and u_j+h_j and u_j, $j\in J$, are the values of $f_j(t)$ on $(q/p, (q+1)/p)$ and $(r/p, (r+1)/p)$, respectively.

Now from (5) and (6) it follows that [4, formula (10)]

$$\int_0^\delta \{\Phi(a + \delta + t, u_i, U_J) - \Phi(a + \delta + t, u_i + h_i, U_J) + \Phi(a + t, u_i + h_i, U_J + H_J) - \Phi(a + t, u_i, U_J + H_J)\}dt \geqq 0. \tag{12}$$

We add to (12) the $(l-1)$ inequalities obtained from (12) by replacing a by $a+\delta, \cdots, a+(l-1)\delta$ and H_J each time by zero. This gives

$$\int_0^\delta \{\Phi(a + l\delta + t, u_i, U_J) - \Phi(a + l\delta + t, u_i + h_i, U_J) + \Phi(a + t, u_i + h_i, U_J + H_J) - \Phi(a + t, u_i, U_J + H_J)\}dt \geqq 0. \tag{13}$$

† In addition, we may assume $h_i \leqq u_i'$, if f_i, g_i are positive.

From the convexity of Φ with respect to u_i it follows (for instance by means of (9)) that

$$\Phi(u_i + h_i) - \Phi(u_i) - \Phi(u_i') + \Phi(u_i' - h_i) \geqq 0$$

if $u_i' \leqq u_i$, $h_i \geqq 0$. Replacing here t by $a + l\delta + t$ and adding to the integrand of (13), we obtain $\Delta \geqq 0$, which completes the proof.

In particular we obtain (compare [4]):

COROLLARY. *If Φ has continuous second derivatives,* (2) *holds if and only if*

$$\frac{\partial^2 \Phi}{\partial u_i \partial u_j} \geqq 0, \qquad \frac{\partial^2 \Phi}{\partial t \partial u_i} \leqq 0, \qquad i, j = 1, \cdots, n. \tag{14}$$

3. We consider some examples. Let $\Phi = \phi(t)\Psi(u_1, \cdots, u_n)$. Then condition (6) reduces to

$$[\Psi(u_i + h_i) - \Psi(u_i)] \int_0^\delta \{\phi(a + \delta + t) - \phi(a + t)\} dt \leqq 0$$

or to

$$[\Psi(u_i + h_i) - \Psi(u_i)][\phi_1(a + 2\delta) - 2\phi_1(a + \delta) + \phi_1(a)] \leqq 0,$$

where $\phi_1(t)$ is an integral of $\phi(t)$. Hence (6) holds if and only if either $\phi(t)$ is a constant, or $\phi(t)$ and $\Psi(u_i)$ for each i are monotone in the opposite sense.

Further examples are $\Phi = F(u_1)$ with a convex F; this gives an inequality of Hardy, Littlewood and Pólya ([3], [2]), and the following two inequalities which appear to be new:

$$\int_0^1 F[f_1(t) + \cdots + f_n(t)] dt \leqq \int_0^1 F[g_1(t) + \cdots + g_n(t)] dt, \tag{15}$$

$$\int_0^1 F[f_1(t) + \cdots + f_n(t)] dt \leqq \int_0^1 F[g_1(t) + \cdots + g_n(t)] dt. \tag{16}$$

In (15), $F(u)$ is a convex function defined for all values of u; in (16), $F(u)$ is convex and increasing for $u \geqq 0$. The inequalities follow at once from Theorem 1, since $\Phi = F(u_1 + \cdots + u_n)$ and $\Phi = F(u_1 \cdots u_n)$ satisfy all conditions (5), (6), and (7).

If we define the relation $f \prec g$ by (3), omitting (4), we obtain similar results. If $f = u$, $g = v$ are constants with $u \leqq v$, then $f \prec g$. Hence Φ in (2) must be increasing in each variable u_i. Together with (5), (6), and (7) this is also sufficient for (2). This is seen from the proof of Theorem 1 if we supplement the transformations T used there by a transformation T' which increases (in the wide sense) each of the values of a step-function f_i. As $f \prec g$ on $(0, 1)$ now implies $f \prec g$ on $(0, x)$, $0 \leqq x \leqq 1$, we may replace the limits of integration 0, 1 in (2) by 0, x. (A similar remark applies to (1) if Φ is increasing.)

We could have required that (2) holds for all $f_i \prec g_i$ (this relation defined by (3) and (4)), dropping the assumption that f_i, g_i are decreasing. This gives, as in Theorem 1, (a), the relation (11) with the h_j now of arbitrary sign. It follows that $\Phi(u_i+h_i)-\Phi(u_i)$ and therefore also $\partial\Phi/\partial u_i$ does not depend on the u_j with $j\neq i$. Integrating we get $\Phi=\phi(t, u_i)+\psi(t, U_J)$. In this way we obtain

THEOREM 2. *A function Φ satisfies* (2) *with f_i, g_i not necessarily decreasing if and only if Φ is of the form*

$$\Phi(t, u_1, \cdots, u_n) = \sum_{i=1}^{n} \Phi_i(t, u_i),$$

where each Φ_i is a function of two variables which satisfies (6) *and* (7).

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A PROOF OF THE IRRATIONALITY OF π

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Assume $\pi=a/b$, a and b integers. Then, with $N=2a$, $\sin N=0$, $\cos N=1$, and $\cos (N/2)=\pm 1$.

If m is zero or a positive integer, then

$$A_m(x) \equiv \sum_{k=0}^{\infty} (-1)^k(2k+1)^m \frac{x^{2k+1}}{(2k+1)!} = P_m(x)\cos x + Q_m(x)\sin x$$

where $P_m(x)$ and $Q_m(x)$ are polynomials in x with integral coefficients. (Proof by induction on m: $A_{m+1}=xdA_m/dx$, and $A_0=\sin x$.)

Thus $A_m(N)$ is an integer for every positive integer m.

If t is any positive integer, then

$$\begin{aligned} B_t(N) &\equiv \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1-t-1)(2k+1-t-2)\cdots(2k+1-2t)}{(2k+1)!} N^{2k+1} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^t - b_1(2k+1)^{t-1} + \cdots \pm b_t}{(2k+1)!} N^{2k+1} \\ &= A_t(N) - b_1A_{t-1}(N) + \cdots \pm b_tA_0(N). \end{aligned}$$

Since all the b_i are integers, $B_t(N)$ must be an integer too.

Now

$$B_t(N) = \sum_{k=0}^{[(t-1)/2]} + \sum_{k=[(t+1)/2]}^{t-1} + \sum_{k=t}^{\infty}.$$

In the first sum, the numerator of each fraction is a product of t consecutive integers, therefore it is divisible by $t!$, and therefore by $(2k+1)!$ since $2k+1 \leqq t$. Thus each term of the first sum is an integer. Each term of the second sum is zero. Thus the third sum must be an integer, for every positive integer t.

This third sum is

$$\sum_{k=t}^{\infty} (-1)^k \frac{(2k-t)!}{(2k+1)!(2k-2t)!} N^{2k+1}$$

$$= (-1)^t \frac{t!}{(2t+1)!} N^{2t+1} \left(1 - \frac{(t+1)(t+2)}{(2t+2)(2t+3)} \frac{N^2}{2!} + \frac{(t+1)(t+2)(t+3)(t+4)}{(2t+2)(2t+3)(2t+4)(2t+5)} \frac{N^4}{4!} - \cdots\right).$$

Let $S(t)$ stand for the sum in the parenthesis. Certainly

$$|S(t)| < 1 + N + \frac{N^2}{2!} + \cdots = e^N.$$

Thus the whole expression is absolutely less than

$$\frac{t!}{(2t+1)!} N^{2t+1} e^N < \frac{N^{2t+1}}{t^{t+1}} e^N < (N^2/t)^{t+1} e^N;$$

for $t > t_0$, this is certainly less than 1.

Therefore necessarily $S(t) = 0$ for every integer $t > t_0$. But this is impossible, because

$$\lim_{t\to\infty} S(t) = 1 - \frac{1}{2^2}\cdot\frac{N^2}{2!} + \frac{1}{2^4}\cdot\frac{N^4}{4!} - \cdots = \cos(N/2) = \pm 1.$$

It can be proved similarly that the natural logarithm of a rational number must be irrational: From $\log(a/b) = c/d$ would follow $e^c = a^d/b^d = A/B$. Then

$$B \cdot \sum_{k=0}^{\infty} \frac{(k-t-1)(k-t-2)\cdots(k-2t)}{k!} c^k$$

would have to be an integer for every positive integer t, and a contradiction results, as before.

CLASSROOM NOTES

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A RULE FOR SOLVING A CERTAIN TYPE OF EXACT DIFFERENTIAL EQUATIONS

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In solving an exact differential equation

$$Mdx + Ndy = 0 \qquad \left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right), \tag{1}$$

it often happens that M or N is a function of the type

$$\sum_{i=1}^{n} \phi_i(x)\psi_i(y). \tag{2}$$

In that case, the following simple rule may be used to obtain the solution of (1).

Suppose, for definiteness, that N is of the form (2). We write $N = N_1 + N_2$, where N_2 is the sum of the terms of N for which $\phi_i(x)$ is a constant. The solution of equation (1) is then given by

$$\int Mdx + \int N_2 dy = C, \tag{3}$$

where C is a constant.

Proof. Since equation (1) is exact, we have $dF = Mdx + Ndy$ and

$$F = \int Mdx + \int \left\{N - \int \frac{\partial M}{\partial y}\,dx\right\} dy = \int Mdx + \int \left\{N - \int \frac{\partial N}{\partial x}\,dx\right\} dy$$

$$= \int Mdx + \int \{N - N_1\}dy = \int Mdx + \int N_2 dy.$$

Similarly, if M is of the form (2) and $M = M_1 + M_2$ where M_2 is the sum of the terms of M for which $\psi_i(y)$ is a constant, then the solution of equation (1) is given by

$$\int Ndy + \int M_2 dx = C.$$

Example: Solve the exact differential equation

$$(\tan y + \sec^2 x)dx + \sec y(\tan y + x \sec y)dy = 0.$$

Observe that $N_2 = \sec y \tan y$. Hence $\int Mdx + \int N_2 dy = C$, becomes $x \tan y + \tan x + \sec y = C$.

REMARKS ON YATES' NOTE "DIFFERENTIATING THE LOGARITHM"

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In the note mentioned (this MONTHLY, February 1954, p. 120) it was shown that the Δ-process works more naturally with $x \ln x$ than with $\ln x$ in that, at a certain stage in the differentiation of $\ln x$ "it is necessary to introduce a carefully selected coefficient." Thence, by differentiating $x \ln x$ as a product the derivative of $\ln x$ is obtained *if it has a derivative*, an assumption omitted in the note. The assumption can, of course, be dispensed with if $x \ln x/x$ be differentiated as a quotient. However if we are willing to assume that $\ln x$ is differentiable the following remarks may be even more natural. If f is a differentiable function defined for $x>0$ such that $f(ax)=f(a)+f(x)$, differentiating gives $af'(ax)=f'(x)$, hence $f'(ax)=f'(x)/a$, thus $f'(a)=c/a$ where $c=f'(1)$. The derivative of $\ln x$ at $x=1$ is $\lim \ln (1+h)^{1/h}$, and no "carefully selected coefficient" is necessary.

TOSSING A COIN

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A basic fact concerning random experiments treated in practically all books on probability and statistics is stated in one of them* in the following words: "If an ordinary coin is rapidly spun several times, and if we take care to keep the conditions of the experiment as uniform as possible in all respects, we shall find that we are unable to predict whether, in a particular instance, the coin will fall 'heads' or 'tails'." The page devoted to a general discussion of this experiment ends with the following remark. "A moment's reflection will show that even extremely small changes in the initial state of the motion must be expected to have a dominating influence on the result. In practice, the initial state will never be exactly known, but only to a certain approximation."

Since a quantitative description of the problem can be achieved in terms of the most elementary mathematics and in about as many words as are required by a general discussion, there is no reason why every beginner should not learn the quantitative details of the case. If the following remarks† should be contained somewhere in the literature, they do not seem to have found their way into the standard books on probability.

We choose a coin of radius 1, lift it from a table, bring it into a vertical position, spin it about its horizontal diameter, and drop it. All that is relevant to the problem occurs in the vertical plane which passes through the center of the lifted coin and is perpendicular to the axis of rotation. At any instant, the intersection of this plane with the coin is a flat rectangle of length 2. Of its two long sides, one may be assumed to be marked H(ead), and the other T(ail).

* Cramér, Mathematical Methods of Statistics, Princeton University Press, 1946, pp. 138 seq.

† They elaborate on an exercise in Menger, Calculus, A Modern Approach, 2nd edition, Illinois Institute of Technology Bookstore, 1953, p. 174. (There, the initial position of the coin is horizontal.) The author is indebted to Prof. Eli Sternberg for suggestions concerning this elaboration.

We thus consider a vertical halfplane, bounded by a horizontal line. A rod of length 2 is lifted and brought into a vertical position (say, with the mark H on right side, and T on the left). Then, within the halfplane, the rod is spun about its center, and dropped with the initial velocity 0. If the rod is assumed to be homogeneous and ideally slender, and the resistance of the air is neglected, the laws of mechanics imply:

A. The distance travelled by the center of the rod is proportional to the square of the time elapsed since the release.

B. The initial angular velocity of the rod, due to the spin, remains constant until the rod touches the bottom line.

Moreover, we make the following (more convenient than realistic) assumptions which neglect the effects of the spin after the first contact between the rod and the bottom.

C. If, when first touching the bottom line, the rod is in a vertical position, then it will remain in this unstable equilibrium and we say that the outcome of the tossing is V(ertical).

D. If case C is not present, then, unless the rod reaches the bottom line in a horizontal position, it will pivot about the first point of contact and come to a definite horizontal rest after traversing the smaller of the two angles that it forms with the bottom line on first contact.

D implies that the mark (H or T) which is on the upper side of the rod at the instant of its first contact will remain on the upper side, and thus determine the outcome of the tossing.

By virtue of A and B, if proper units of time and distance are chosen, t units of time after the release of the rod, the altitude of its lowest point is

$$h(t) = a - t^2 - |\cos ct| \tag{1}$$

units of length, if a denotes the initial altitude of the center of the rod and c the angular velocity. Clearly, $a \geqq 1$.

The rod is in a vertical position at $t = k\pi/c$ for $k = 0, 1, \cdots$. If $c > 0$, that is, if the spin is counterclockwise, then, since at the initial instant the mark H is assumed to be on the right side, H is on top during the intervals

$$H^+: \quad 2m\pi/c < t < (2m+1)\pi/c, \qquad m = 0, 1, \cdots,$$

and T is on top during the intervals

$$T^+: \quad (2m-1)\pi/c < t < 2m\pi/c, \qquad m = 1, 2, \cdots.$$

If $c < 0$, the situation is reversed, that is, $H^- = T^+$ and $T^- = H^+$.

Let k be that integer for which

$$k^2\pi^2/c^2 + 1 \leqq a < (k+1)^2\pi^2/c^2 + 1. \tag{2}$$

If $t < k\pi/c$, then, by (1),

$$h(t) = a - t^2 - |\cos ct| > a - k^2\pi^2/c^2 - 1$$

and thus $h(t)>0$, by (2). Therefore, since h is a continuous function of t, the instant t_1 of the first contact of the bar with the bottom (*i.e.*, the smallest solution of the equation $h(t)=0$) satisfies the condition

$$k\pi/c \leqq t_1 < (k+1)\pi/c. \tag{3}$$

Conversely, (3) implies (2) since, if the altitude lay in an interval different from (2), t_1 would lie in an interval different from (3).

By assumption C, the outcome is V if in (3), and therefore in (2), the equality sign holds. In case of inequalities in (2) and (3), by assumption D, the outcome, provided $c>0$, is H or T depending on whether k in (2) is even or odd and t_1 accordingly belongs to H^+ or T^+. The situation is reversed if $c<0$.

The initial state of the rod, and therefore the outcome of the tossing, is determined by the numbers a and c. Each possible initial state can be represented by a point in an (a, c) halfplane $(a \geqq 1)$. The points on the curves

$$C_k: \quad a = k^2\pi^2/c^2 + 1 \qquad (k = 1, 2, \cdots)$$

and on the lines

$$C_0: \quad c = 0 \text{ and } a = 1$$

represent initial states resulting in the outcome V. The states in the boomerang shaped regions between C_{2m} and C_{2m+1} for $c>0$ and between C_{2m-1} and C_{2m} for $c<0$ result in H. The states in the other regions result in T.

Suppose now that the initial state is not accurately known and all that can be ascertained is that the initial altitude lies between a and $a+\alpha$, and the angular velocity between c and $c+\gamma$. If a and c satisfy (2), the number $(a-1)c^2$ is confined to the interval between $k^2\pi^2$ and $(k+1)^2\pi^2$. If

$$(a+\alpha-1)(c+\gamma)^2 > (k+1)^2\pi^2, \tag{4}$$

then initial conditions which are indistinguishable from (a, c) lead to a different outcome, *and the outcome of the tossing is unpredictable.**

For instance, this is the case if $\alpha=0$ (that is to say, if the altitude can be accurately determined), $c>0$, and

$$\gamma > \frac{(k+1)\pi}{\sqrt{a-1}} - c > 0.$$

Another example is the case where $\gamma=0$ (that is to say, the angular velocity can be accurately determined) and

$$\alpha > \frac{(k+1)^2\pi^2}{c^2} - a + 1 > 0.$$

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* Besides the inaccuracies in determining a and c, the facts that the initial verticality of the coin and the horizontality of its axis of rotation can be only approximately materialized add to the unpredictability of the outcome of the tossing.

ON THE ITERATION OF CONTINUOUS NON-INCREASING FUNCTIONS

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Let $f(x)$ be continuous and non-increasing, and let x_0 be an arbitrary real number. We consider sequences obtained by iteration of $f(x)$, *i.e.*, sequences defined by $x_n=f(x_{n-1})$ $(n=1, 2, \cdots)$. Theorems are presented on the behavior of such sequences. Included among these theorems are a useful sufficient condition for convergence and theorems about the set of all x_0 for which x_n converges. In all the theorems x_n and $f(x)$ are as defined above.

THEOREM 1. *If $x_n \to x$ then $x=f(x)$.*

Proof. Since $f(x)$ is continuous, $x=\lim f(x_n)=f(\lim x_n)=f(x)$.

THEOREM 2. *For all x_0 for which x_n converges, $\lim x_n$ has the same value.*

Proof. Since $f(x)$ is non-increasing, there can be at most one solution of the equation $x=f(x)$.

THEOREM 3. *Each sequence satisfies one of the following sets of relations:*

a. (1) $x_0 \leqq x_2 \leqq \cdots \leqq x_{2n} \leqq \cdots \leqq x_{2n+1} \leqq \cdots \leqq x_3 \leqq x_1$

(2) $x_1 \leqq x_3 \leqq \cdots \leqq x_{2n+1} \leqq \cdots \leqq x_{2n} \leqq \cdots \leqq x_2 \leqq x_0$.

b. (1) $\cdots \leqq x_{2n} \leqq \cdots \leqq x_2 \leqq x_0 \leqq x_1 \leqq x_3 \leqq \cdots \leqq x_{2n+1} \leqq \cdots$

(2) $\cdots \leqq x_{2n+1} \leqq \cdots \leqq x_3 \leqq x_1 \leqq x_0 \leqq x_2 \leqq \cdots \leqq x_{2n} \leqq \cdots$.

Proof. This is an immediate consequence of the non-increasing behavior of $f(x)$. Thus it is seen that the subsequences x_{2n} and x_{2n+1} are always monotonic.

THEOREM 4. *Let x_n be bounded, $\underline{x}=\underline{\lim}\, x_n$, and $\bar{x}=\overline{\lim}\, x_n$. Then $\bar{x}=f(\underline{x})$ and $\underline{x}=f(\bar{x})$.*

Proof. From Theorem 3 it as follows that either

$$\underline{x} = \lim x_{2n} \quad \text{and} \quad \bar{x} = \lim x_{2n+1}$$

or

$$\underline{x} = \lim x_{2n+1} \quad \text{and} \quad \bar{x} = \lim x_{2n}.$$

Taking limits in $x_{2n}=f(x_{2n-1})$ and $x_{2n+1}=f(x_{2n})$ gives the desired results.

THEOREM 5. *A sufficient condition for the convergence of x_n is that it be bounded and that the only solutions of the equations $y=f(x)$ and $x=f(y)$ which lie within these bounds have $y=x$.*

Proof. By Theorem 4, $\underline{\lim}\, x_n=\overline{\lim}\, x_n$.

The condition on the solutions of $y=f(x)$ and $x=f(y)$ can be interpreted geometrically. It says that the curve $y=f(x)$ and its mirror image through the

line $y=x$ intersect only on the line $y=x$ (at least within the bounds of x_n). Sometimes (*e.g.*, if $f(x)=e^{-x}$) it is readily seen, either by drawing the curves or by simple analysis, that the curves have exactly one point of intersection, which must lie on $y=x$, since (y_0, x_0) is a point of intersection if (x_0, y_0) is.

THEOREM 6. *Let the curves $y=f(x)$ and $x=f(y)$ have exactly one point of intersection. Furthermore let $f(x)$ be either bounded from below or bounded from above. Then x_n converges for all x_0.*

Proof. Suppose $-\infty < a \leqq f(x)$ $(-\infty < x < +\infty)$.

Then $a \leqq x_n \leqq f(a)$ $(n \geqq 2)$. Hence by Theorem 5 x_n converges. A similar argument holds if $f(x) \leqq b < \infty$ $(-\infty < x < +\infty)$. Thus since $e^{-x} \geqq 0$, $f(x)=e^{-x}$ yields a convergent sequence for all x_0.

THEOREM 7. *Let x_n converge when $x_0=a$. Then for all x_0 such that min $(a, f(a)) \leqq x_0 \leqq$ max $(a, f(a))$, x_n converges.*

Proof. Iteration gives the desired result.

THEOREM 8. *Let C be the set of all x_0 for which x_n converges. Then C is non-vacuous and includes all x_0 satisfying inf $C < x_0 <$ sup C.*

Proof. Choose any x_0. Either $x_0 \leqq f(x_0)$ and $x_1 \geqq f(x_1)$ or $x_0 \geqq f(x_0)$ and $x_1 \leqq f(x_1)$. Thus there exists a unique p such that min $(x_0, x_1) \leqq p \leqq$ max (x_0, x_1) and $p=f(p)$. Clearly p belongs to C. Now suppose that inf $C < x_0 <$ sup C and $x_0 \neq p$. If $x_0 < p$, choose a belonging to C such that $a < x_0$; if $x_0 > p$, choose a belonging to C such that $a > x_0$. In either case, min $(a, f(a)) \leqq x_0 \leqq$ max $(a, f(a))$. Hence by Theorem 7 x_0 belongs to C.

THEOREM 9. *Let $\alpha = inf\ C$ and $\beta = sup\ C$. Then either $\alpha = -\infty$ and $\beta = +\infty$ or α and β are both finite with $\alpha = f(\beta)$ and $\beta = f(\alpha)$.*

Proof. Suppose α is finite. Clearly $\alpha \leqq p = f(p)$. Hence $f(\alpha) \geqq \alpha$. Now consider the points $f(\alpha-\epsilon)$ $(\epsilon > 0)$. These points do not belong to C; otherwise $\alpha - \epsilon$ would. Since $f(\alpha) = f(\alpha - 0) \leqq f(\alpha - \epsilon)$, we must by Theorem 8 have $\beta \leqq f(\alpha)$. On the other hand since there are points arbitrarily close to α which belong to C, there must be points arbitrarily close to $f(\alpha)$ which belong to C. Therefore $\beta \geqq f(\alpha)$. Thus $\beta = f(\alpha)$. Similarly if β is finite, then $\alpha = f(\beta)$.

THEOREM 10. *Let the curves $y=f(x)$ and $x=f(y)$ have exactly one point of intersection. Then C is either all the reals or just $p=f(p)$.*

Proof. This is an immediate consequence of Theorem 9. That p can be the only point of C can be seen by taking $f(x) = -2x$. By taking $f(x) = -x/2$, it is seen that C can be all the reals without $f(x)$ being bounded.

So far we have assumed that $f(x)$ is non-increasing for all real x. Obviously some of our results are valid if $f(x)$ is non-increasing over an interval containing x_n. For example, consider $f(x) = \cos x$. Irrespective of x_0, $0 \leqq x_n \leqq 1$ $(n \geqq 2)$.

Since the cosine is non-increasing over this range, we may write the equations:

$$\bar{x} = \cos \underline{x}$$

$$\underline{x} = \cos \bar{x}$$

$$\bar{x} - \underline{x} = 2 \sin \frac{\bar{x} + \underline{x}}{2} \sin \frac{\bar{x} - \underline{x}}{2}.$$

Assume $\bar{x} \neq \underline{x}$. Then we may write

$$\frac{\sin \dfrac{\bar{x} - \underline{x}}{2}}{\left(\dfrac{\bar{x} - \underline{x}}{2}\right)} = \frac{1}{\sin \dfrac{\bar{x} + \underline{x}}{2}}. \tag{1}$$

Since $0 < [(\bar{x}+\underline{x})/2] < 1 < (\pi/2)$, the right hand side of (1) is greater than one. But the left hand side is less than one. Therefore x_n converges for all x_0.

ON AN APPLICATION OF THE VANDERMONDE DETERMINANT

W. V. PARKER, Alabama Polytechnic Institute

In this MONTHLY for December 1953, Harley Flanders gave a solution of the following problem. Let $x_1, x_2, \cdots, x_n$ be n numbers such that $\sum x_i = \sum x_i^2 = \cdots = \sum x_i^n = 0$; to prove that $x_1 = x_2 = \cdots = x_n = 0$. He used the Vandermonde determinant and an induction process.

The following solution, which uses the Vandermonde determinant but avoids the induction process, may be of interest. Let the solution be such that n_i of the x's are equal to $a_i \neq 0$, $i = 1, 2, \cdots, k$, and $a_i \neq a_j$, $i \neq j$, and the remaining x's are zero. Then we have $\sum n_i a_i = \sum n_i a_i^2 = \cdots = \sum n_i a_i^n = 0$. The first k of these equations in $n_1, n_2, \cdots, n_k$ constitute a homogeneous system whose determinant is $a_1 \cdot a_2 \cdot \cdots \cdot a_k \cdot V(a_1, a_2, \cdots, a_k)$ where $V(a_1, a_2, \cdots, a_k)$ is a nonvanishing Vandermonde determinant. Hence, $n_i = 0$, $i = 1, 2, \cdots, k$, and $x_1 = x_2 = \cdots = x_n = 0$ is the only solution.

REMARK ON THE GENERAL POLYNOMIAL OF THE SECOND DEGREE

LEONARD GILLMAN, Purdue University

In a previous note (this MONTHLY, vol. 61, 1954, p. 191), F. W. Perkins points out that in establishing the invariance under rotation of $B^2 - 4AC$ (*etc.*) in the general polynomial of the second degree, the labor can be considerably reduced by employing the *derivatives* of the new coefficients $a, b, \cdots$ (with respect to the rotation angle θ). The method presented avoids the calculation of b^2 and of $4ac$, but it does require the long computation of the coefficients $a, b, \cdots$ themselves.

Here is an alternate procedure that avoids both. Let x, y denote the old coordinates, and u, v the new. Then

$$(1) \quad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = au^2 + buv + cv^2 + du + ev + f,$$

where

$$(2) \quad u = x\cos\theta + y\sin\theta, \qquad v = -x\sin\theta + y\cos\theta.$$

Using primes to denote derivatives with respect to θ, we have, from (2),

$$(3) \quad u' = v, \qquad v' = -u.$$

We compute the derivative of the right side of (1). Since (1) is an identity, this derivative is zero. With the substitutions (3), we get (after collecting terms),

$$(4) \quad (a' - b)u^2 + (b' + 2a - 2c)uv + (c' + b)v^2 + (d' - e)u + (e' + d)v + f' = 0.$$

Since (4), too, is an identity, all the coefficients are zero. Thus $a' = b$, *etc.*, and, as before, $(b^2 - 4ac)' = 0$.

If the only invariants of concern are those involving the first three coefficients, one may prefer to proceed by first reducing (1) to

$$(5) \quad Ax^2 + Bxy + Cy^2 = au^2 + buv + cv^2$$

(which is an immediate consequence of (1) and (2)).

A disadvantage in all this is that some additional work must be done in order to derive the formula $\cot 2\theta = (A - C)/B$ for removing the cross-term. One way is to set $b = 0$ in (5) and then substitute various combinations of 1's and 0's for x and y (or for u and v). Being derivative-happy at the moment, however, we cannot resist pointing out the following. From the formulas $b' = 2(c - a)$, $(c - a)' = -2b$, we obtain $b'' = -4b$. The student does not know how to solve this differential equation, but he can verify that

$$(6) \quad b = k_1 \sin 2\theta + k_2 \cos 2\theta$$

is a solution. Differentiation in (6) yields

$$(7) \quad c - a = \tfrac{1}{2}b' = k_1 \cos 2\theta - k_2 \sin 2\theta.$$

Since $a = A$, $b = B$ and $c = C$ when $\theta = 0$, we see that $k_2 = B$ and $k_1 = C - A$. Thus $b = (C - A)\sin 2\theta + B\cos 2\theta$, as desired.

COMPLEX ROOTS OF AN INTEGRAL RATIONAL EQUATION

D. TRIFAN, University of Arizona

The following note concerns a method of proving the theorem that an integral rational equation with real coefficients has complex roots occurring in conjugate pairs.

If $x = a + bi$ is a root of the integral rational equation $f(x) = 0$, then as a result of the factor theorem

$$f(x) = [x - (a + bi)]Q(x).$$

Separating $Q(x)$ into its real and imaginary parts we have

$$f(x) = [x - (a + bi)][F(x) + iG(x)].$$

Since $f(x)$ has real coefficients the imaginary part must be zero *i.e.*

$$xG(x) - aG(x) - bF(x) = 0$$

or

$$F(x) = \frac{(x - a)G(x)}{b}.$$

Therefore

$$f(x) = [x - (a + bi)]\left(\frac{(x - a)G(x)}{b} + iG(x)\right)$$

$$= [x - (a + bi)]\left(\frac{x - (a - bi)}{b}\right)G(x).$$

Thus

$$f(a - bi) = 0.$$

TWO THEOREMS ON GROUPS

R. V. Andree and G. M. Petersen, University of Oklahoma

Theorem 1. *If G is a group having more than two elements, then there exist two distinct elements a, b, neither of which is the identity, such that $ab = ba$.*

If there exists an element $x \in G$ which is not self-inverse, $x \neq x^{-1}$, then let $a = x$, $b = x^{-1}$.

If for every $x \in G$, $x = x^{-1}$, then for all a, $b \in G$, $(ab)\cdot(ba) = a(bb)a = aa = e$. Thus, $(ab)^{-1} = ba$. Since $x = x^{-1}$ by assumption, $(ab)^{-1} = ab$. Thus, $ab = ba$ for all a, $b \in G$.

The latter paragraph suffices to prove theorem 2.

Theorem 2. *A group in which each element is self-inverse is necessarily commutative.*

Theorem 1 has proved useful for classroom examination purposes. It is simple enough to be proved by an average student and can be "half proved" by the less alert, who may fail to consider the case covered by Theorem 2.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1136. *Proposed by W. A. Bowers, University of North Carolina*

An elementary physics text, wishing to avoid using calculus, and also hoping (unsuccessfully!) to avoid the appearance of pulling the answer out of a hat, gives the following recipe for calculating the work done in moving a positive unit charge from r_1 to r_2 against the attraction of a negative unit charge at the origin. "Since the force varies with distance, we cannot simply multiply 'the force' by the distance; instead we must take the geometric mean of the initial and final values of the force, and multiply it by the distance." Noting with amazement that this does indeed work, we ask: what is the most general function for which this is true, that is, for which the average value over an arbitrary interval equals the geometric mean of the values at the end points?

E 1137. *Proposed by Simon Green, Philander Smith College, Arkansas*

Let $N=1+1/2+1/3+\cdots+1/n$. Prove that $e^N>n+1$.

E 1138. *Proposed by J. P. Ballantine, University of Washington*

For any triangle prove that: (1) if $B=2A$, then $b^2=a^2+ac$, (2) if $B=3A$, then $b^3-ab^2-a^2b-ac^2+a^3=0$.

E 1139. *Proposed by N. A. Court, University of Oklahoma*

Three collinear points P, Q, R are marked on the sides BC, CA, AB of a triangle ABC. Starting with an arbitrary point X of the line BC, the following points are constructed successively:

$$Y=(XR, CA), \qquad Z=(YP, AB), \qquad X'=(ZQ, BC);$$
$$Y'=(X'R, CA), \qquad Z'=(Y'P, AB), \qquad X''=(Z'Q, BC).$$

Show that points X and X'' coincide.

E 1140. *Proposed by Albert Wilansky, Lehigh University*

Let sequences $\{x_n\}$, $\{y_n\}$ be called related if $\sum_{k=1}^{n}(x_k-y_k)$ is a bounded function of n. For example, $\{(-1)^n\}$ and $\{0\}$ are related. Is there a bounded sequence not related to any convergent sequence?

SOLUTIONS

The Intersection of Two Secants

E 1106 [1954, 194]. *Proposed by C. I. Lubin, University of Cincinnati*

Two non-parallel, non-coincident lines which cut the circle $|z| = r$ in the points a, b and c, d respectively, where a, b, c, d are complex numbers not necessarily all different, intersect in the point z given by

$$z = (a^{-1} + b^{-1} - c^{-1} - d^{-1})/(a^{-1}b^{-1} - c^{-1}d^{-1}).$$

Solution by J. R. Hatcher, B. F. Peery, and Ineatha Walker, Fisk University. Since $a\bar{a} = b\bar{b} = c\bar{c} = d\bar{d} = r^2$, the equations of the lines are $r^2z + ab\bar{z} = r^2(a+b)$ and $r^2z + cd\bar{z} = r^2(c+d)$. Hence the intersection point z is given by

$$z = [cd(a + b) - ab(c + d)]/(cd - ab),$$

so that dividing numerator and denominator by $abcd$ gives the desired result.

Also solved by Hüseyin Demir, F. A. Ficken, Norman Greenspan, Louisa Grinstein, J. R. Jackson, M. W. Oliphant, J. H. Simester, F. Underwood, and the proposer.

A Pencil of Planes Associated with a Tetrahedron

E 1107 [1954, 194]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

On the edges AB, AC, AD of a tetrahedron $ABCD$ are marked points M, N, P such that $AB = nAM$, $AC = (n+1)AN$, $AD = (n+2)AP$. Show that the plane MNP contains a fixed line as n varies.

I. *Solution by Hüseyin Demir, Zonguldak, Turkey.* From the relations it is evident that the ranges of points $[M]$ and $[P]$ are projective. But since A is a self-corresponding element, the projectivity is a perspectivity. Hence MP is on a fixed point P'. Similarly MN is on a fixed point N'. Hence the plane MNP is on the fixed line $P'N'$.

II. *Solution by C. P. Pinzka, Princeton, N. J.,* Let X be the fourth vertex of the parallelogram determined by AB and BC. Since MX divides AC in the same ratio as N, MN must pass through X. Similarly, NP passes through Y, the fourth vertex of the parallelogram determined by AC and CD. These conclusions hold for any n. It follows that plane MNP must pass through the fixed line XY.

Also solved by Paul Berry, J. R. Jackson, Josef Langr, D. C. B. Marsh, D. B. Mumford, C. S. Ogilvy, J. A. Tierney, F. Underwood, Chih-yi Wang, and the proposer.

Sum of Cubes as Difference of Two Squares

E 1108 [1954, 194]. *Proposed by René Bloch, Humanistisches Gymnasium, Basle, Switzerland*

Let n, k, $a+1$ be three positive integers which are not all odd. Express $\sum_{i=0}^{k} (n+ai)^3$ as a difference of two integer squares.

Solution by the Proposer. We have

$$\sum_{i=0}^{k}(n+ai)^3 = (k+1)n^3 + 3n^2a\binom{k+1}{2} + na^2k(2k+1)(k+1)/2 + a^3\binom{k+1}{2}^2$$

$$= \left[n(k+1) + \binom{k+1}{2}a\right]\left[\binom{n+ak+1}{2} + \binom{n}{2} + k\binom{a}{2}\right].$$

Denoting the first factor by $p-q$ and the second by $p+q$ we find

$$p = \binom{n+1}{2} + nk(a+1)/2 + \binom{a+1}{2}\binom{k+1}{2},$$

$$q = \binom{n}{2} + nk(a-1)/2 + \binom{a}{2}\binom{k+1}{2}.$$

Since n, k, $a+1$ are not all odd it follows that p and q are integers, and

$$\sum_{i=0}^{k}(n+ai)^3 = p^2 - q^2.$$

Also solved by A. R. Hyde, M. J. Pascual, and Walter Penney.

Reversed Squares

E 1109 [1954, 194]. *Proposed by Erich Michalup, Caracas, Venezuela*

Find numbers so that their squares, when reversed, are the squares of the reversed numbers.

Contributions by W. V. Gamzon, Vern Hoggatt, Edgar Karst, D. C. B. Marsh, and A. P. Rhodes.

It can be shown that suitable numbers can contain no digits other than 0, 1, 2, 3, that they must not begin or end with 0, that if one end is 3 the next place cannot be 2, that when such a number is squared there must be no carry-overs in the addition part of the multiplication algorithm. With these restrictions in mind it is easy to construct numbers of the desired sort.

An Ideal Cryptarithm

E 1110 [1954, 194]. *Proposed by Edgar Karst, Independence, Missouri*

An *ideal* cryptarithm has only one solution and involves all ten digits. Solve the following ideal addition cryptarithm based upon the name of the daily newspaper, Hannoversche Presse:

$$\begin{array}{r} H\ A\ N\ N\ O\ V \\ E\ R\ S\ C\ H\ E \\ \hline P\ R\ E\ S\ S\ E \end{array}.$$

Solution by M. A. Kirchberg, Eastern Illinois State College. Obviously $V=0$, $A=9$, $H+E+1=P$, whence we must have $NNO+SCH=1ESS$. Now $N+C \geqq S-1$, $10+E \geqq N+S$, $C+E+11 \geqq 2S$, $S=7$ or 8; $14 \geqq O+H \neq S+10$, $14 \geqq N+C \neq S+10$, $10+E=N+S$, $3 \leqq N \leqq 7$. Upon checking the ten remaining combinations of S and N it is readily seen that the unique solution is $293360+178521=471881$.

Also solved by P. M. Berry, R. L. Caskey, J. E. Darraugh, Hüseyim Demir, William Douglas, Jean Gamzon, A. St. C. G. Grant, Louisa Grinstein, J. D. Haggard, W. P. Hennessy, Vern Hoggatt, A. R. Hyde, Virginia Johnson, Diane Kocher, Sidney Kravitz, D. C. B. Marsh, Don Mattis, J. H. Means, D. B. Mumford, David Muskat, J. B. Muskat, Walter Penney, Elizabeth de Picciotto, G. G. Roberts, Azriel Rosenfeld, David Rothman, C. W. Thomson, C. W. Trigg, W. D. Ward, Monica Wyzalek, the proposer, and a solver with an illegible signature. Late solution by W. T. Grant.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4608. *Proposed by José L. Massera, Instituto de Matematica y Estadistica, Montevideo, Uruguay*

Let a_i, b_i $(i=0, 1, \cdots, 2n)$ be real numbers satisfying the following assumptions:

(a) $a_i + a_{i+1} \geqq 0$ for $i = 0, 1, \cdots, 2n-1$;

(b) $a_{2i+1} \leqq 0$ for $i = 0, 1, \cdots, n-1$;

(c) $\sum_{k=2p}^{2q} b_k > 0$ for $0 \leqq p \leqq q \leqq n$.

Prove that

$$\sum_{i=0}^{2n} (-1)^i a_i b_i \geqq 0$$

and that the equality sign holds only if $a_i = 0$ for all i.

4609. *Proposed by Vern Hoggatt, San Jose State College, California*

Let $q_k(F_n)$ be the kth digit (from the right) of F_n, the nth Fibonacci number ($F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$). Let s be defined such that $q_k(F_n) = 0$ for $n = 0, 1, 2, \cdots, s$ but $q_k(F_{s+1}) \neq 0$. (1) Show that the sequence $q_k(F_n)$ has at least another run of $s+1$ consecutive zeros. (2) Prove or disprove: the sequence $q_k(F_n)$ has never more than $s+1$ consecutive zeros.

4610. *Proposed by Joseph Lehner, Los Alamos Scientific Laboratory*

Show that for almost all real, irrational θ, the series

$$\sum_{n=1}^{\infty} \log \left| \sin n\pi\theta \right| \cdot x^n$$

converges in the unit circle.

4611. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given five spheres, if one of them is orthogonal to the four others, then the centers of the four are the vertices of an orthocentric tetrahedron whose orthocenter coincides with the center of the fifth sphere.

4612. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Let a_0, a_1 be arbitrary and $a_n = a_{n-1} + a_{n-2}/n(n-1)$ for $n > 1$. Find

$$\lim_{n \to \infty} a_n.$$

SOLUTIONS

Systems of Numeration Having Squares of Special Form

4543 [1953, 423]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine systems of numeration in which there are one or more four-digit squares of the form *aabb* and such that $a+b$ equals the sum of the digits of the square root.

Partial Solution by W. J. Blundon, Memorial University of Newfoundland. The assumption is that, if x is the base of the system of numeration, then integers p, q exist such that

$$(ax^2 + b)(x + 1) = (px + q)^2,$$

where a, b, p, $q < x$ and $a+b = p+q$. Two particular cases give interesting sets of solutions:

Case I. Let $a+b=p+q=x+1$.
Elimination of b, q from the given equation gives

$$(ax - a + 1)(x + 1)^2 = (px + x - p + 1)^2. \tag{1}$$

Therefore $x+1$ divides $px+x-p+1$, so that $x+1$ divides $2p$. Since $p<x$, it follows that $x+1=2p$. Substituting in (1) we have $p=2a-1$, $q=2a-1$, $x=4a-3$, $b=3a-2$.

Case II. Let $a+b=p+q=x-1$. Then

$$(ax + a + 1)(x + 1) = (x - 1)(p + 1)^2. \tag{2}$$

Therefore $4a+2=m(x-1)$, where $m=1, 2, 3, 4$ (since $a \leqq x-1$). (2) may now be written

$$(4a + 2m + 1)^2 - m(2p + 2)^2 = 1.$$

If m is 1 or 4, there is no solution in integers. If m is 2 or 3, we have Pell's equation. If m is 3, solutions never yield integral values of x, but if m is 2 the equation becomes

$$(4a + 5)^2 - 2(2p + 2)^2 = 1$$

the smallest solution of which is $a=3$, $p=5$, successive values being given by the recurrence relations

$$a' = 17a + 12p + 32, \qquad p' = 24a + 17p + 46.$$

That the above two cases do not give the complete solution is shown by the examples: $x=97$, $a=34$, $b=94$, $p=57$, $q=71$; $x=161$, $a=148$, $b=142$, $p=154$, $q=136$.

Also partially solved by R. Venkatachalam Iyer, S. Parameswaran, and the Proposer.

Probability of an Odd Determinant

4544 [1953, 423]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

The elements of a determinant are arbitrary integers. Determine the probability that the value of the determinant is odd.

Solution by E. H. Cutler, Lehigh University. Let P_n be the required probability where n, the order of the determinant, is fixed. Then P_n is also the probability that a determinant of elements 0 and 1 shall have its value $\equiv 1 \pmod 2$, and we now consider only values (mod 2) of n-rowed determinants with elements 0 or 1.

The probability that the first row shall be $(0, 0, \cdots, 0)$ is 2^{-n} and the probability that such a determinant have value 1 is 0. If the first row is $(1, 0, 0, \cdots, 0)$, the probability of the value 1 is P_{n-1}.

If the first row is $(1, x_2, \cdots, x_n)$, there is an elementary transformation re-

ducing each such determinant to one with first row $(1, 0, \cdots, 0)$, and this transformation is its own inverse (mod 2). Thus there exists a 1-1 value-preserving correspondence between the set of all determinants with the one first row and the set of all those with the other first row. Since the order of columns is immaterial, the probability that a determinant with given first row have the value 1 is P_{n-1} unless the first row is $(0, 0, \cdots, 0)$. Hence

$$P_n = (1 - 2^{-n})P_{n-1} = (1 - 2^{-1})(1 - 2^{-2}) \cdots (1 - 2^{-n})$$

since P_1 is $1/2$.

Solved also by W. J. Blundon, Leonard Carlitz, N. J. Fine, Harley Flanders, D. S. Greenstein, D. L. Johnson, D. N. Mesner, G. E. Raynor, E. H. Umberger, and the Proposer.

Editorial Note. As noted by several solvers, the present problem is merely the case $m = 2$ of a result obtained by N. J. Fine and Ivan Niven. See their paper, The probability that a determinant be congruent to a (mod m), *Bulletin of the American Mathematical Society*, vol. 50, 1944, pp. 89–93.

A Divergent Series with a Special Property

4545 [1953, 423]. *Proposed by G. G. Lorentz, Wayne University, Detroit*

Let us say that a series, $\sum v_k$, $v_k \geqq 0$, is dominated by the series $\sum u_n$, $u_n \geqq 0$ if $v_k \geqq u_{n_k}$, $k = 1, 2, \cdots$ for a sequence $n_1 < n_2 < \cdots$ of indices n. Construct a divergent series $\sum u_n$ such that each dominated series with decreasing terms is convergent.

Solution by L. A. Ringenberg, Eastern Illinois State College. For $m = 0, 1, 2, \cdots$, let n_m be the smallest integer such that $\sum_{i=1}^{n_m} 1/i \geqq 2^m$. Let $\sum u_k$ be the series which results when the parentheses are removed from the following series:

$$1 + \frac{1}{2}\left(\frac{1}{4} + \frac{1}{3} + \frac{1}{2} + 1\right) + \frac{1}{4}\left(\frac{1}{n_2} + \frac{1}{n_2 - 1} + \cdots + 1\right) + \cdots + \frac{1}{2^m}\left(\frac{1}{n_m} + \frac{1}{n_m - 1} + \cdots + 1\right) + \cdots . \tag{1}$$

Note that (1) dominates the series $1+1+1+\cdots$; hence (1) and also $\sum u_k$ are divergent. Note also that the terms in $\sum u_k$ may be grouped into blocks of consecutive terms (as suggested by the form of (1)) such that the terms within each block increase with the subscript. Let $\sum v_k$ be any series of nonnegative terms, terms nonincreasing as k increases, dominated by $\sum u_k$. Consider the terms in $\sum v_k$ which are dominated by terms in the block

$$\frac{1}{2^m}\cdot\frac{1}{n_m} + \frac{1}{2^m}\cdot\frac{1}{n_m - 1} + \cdots + \frac{1}{2^m}\cdot$$

There may be n_m terms, each one not exceeding $1/2^m n_m$; or there may be $n_m - 1$

terms, each one not exceeding $1/2^m(n_m-1)$; etc. Hence the sum of these terms of the v_k series does not exceed $1/2^m$ and therefore $\sum v_k \leqq 2$.

Also solved by George Piranian and the Proposer.

An Improper Integral

4546 [1953, 423]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

Evaluate

$$I = \int_0^\infty \frac{\sin\dfrac{mx}{2}\sin\dfrac{nx}{2}\sin\dfrac{m+1}{2}x\sin\dfrac{n+1}{2}x}{x^2\sin^2\dfrac{x}{2}}\,dx$$

where m and n are integers and $m \geqq n$.

Solution by J. V. Whittaker, University of California, Los Angeles. This integral is merely

$$\int_0^\infty \frac{1}{x^2}\sum_{r=1}^{m}\sum_{s=1}^{n}\sin rx\sin sx\,dx.$$

Integrating by parts and applying the sum and difference formulas, we obtain a known form:

$$\int_0^\infty \frac{\sin rx\sin sx\,dx}{x^2} = \int_0^\infty \frac{r\cos rx\sin sx + s\sin rx\cos sx}{x}\,dx$$
$$= \pi s/2 \qquad (r \geqq s).$$

Summing over r and s, we find the value of I to be $(\pi/12)n(n+1)(3m-n+1)$.

Also solved by A. D. Anderson, B. A. Fleishman, Emil Grosswald, L. A. Ringenberg, O. E. Stanaitis, Chih-yi Wang, and the Proposer.

A Special Case of Familiar Inequalities

4547 [1953, 424]. *Proposed by H. S. Shapiro, New York University*

Let $0 \leqq a_i < 1$, $i = 1, \cdots, n$ and put $A = \sum a_i$. Prove

$$\sum_{i=1}^{n}\frac{a_i}{1-a_i} \geqq \frac{nA}{n-A},$$

equality occurring only if all the a_i are equal.

I. *Solution by L. A. Ringenberg, Eastern Illinois State College.*

Let $b_i = 1-a_i$, $B = \sum b_i$, $B_{ij} = b_i/b_j + b_j/b_i$. Then

$$B_{ij} = \frac{b_i^2 + b_j^2}{b_i b_j} \geqq 2,$$

$$\sum_{i=1}^{n} \frac{B}{b_i} = n + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} B_{ij} \geqq n + 2\sum_{i=1}^{n-1} (n - i) = n^2,$$

whence, after dividing by B,

$$\sum_{i=1}^{n} \left(\frac{1}{b_i} - 1\right) \geqq \frac{n(n - B)}{B},$$

which is the required inequality for $a_i < 1$. Equality holds only if every $B_{ij} = 2$, only if all the b_i are equal, only if all the a_i are equal. If one or more of the $a_i = 1$, the inequality may be justified in the sense that, in the extended real number system, if the right member is $+\infty$ then so is the left member.

II. *Solution by Hermann von Schelling, Groton, Connecticut.* For $0 < b_i$, by the well known relation between the harmonic and arithmetic mean we have

$$\frac{1}{n} \sum_{i=1}^{n} \frac{1}{b_i} \geqq \frac{1}{\sum b_i/n}.$$

Putting $b_i = 1 - a_i$, and taking $a_i < 1$, this becomes

$$\frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 - a_i} \geqq \frac{n}{n - A},$$

which reduces easily to the required inequality.

III. *Solution by Robert Frucht, Technical University Santa Maria, Valparaiso, Chile.* The inequality that has to be proved is only the special case: $m_1 = m_2 = \cdots = m_n = 1$, $f(x) = x/(1-x)$, $a = 0$, $b = 1$, of the following more general theorem:* If, for $a \leqq x < b$, $f(x) \geqq 0$ and $f''(x) > 0$, then

$$\frac{\sum m_i f(a_i)}{\sum m_i} \geqq f\left(\frac{\sum m_i a_i}{\sum m_i}\right)$$

if $a \leqq a_i < b$ and $m_i > 0$ for $i = 1, 2, \cdots, n$; equality occurs only if all the a_i are equal.

Also solved by T. M. Apostol, W. J. Blundon, D. G. Duncan, W. B. Fulks and P. G. Kirmser, Harry Furstenberg, H. E. Goheen, D. S. Greenstein, J. W. Hardy, Jr., Vern Hoggatt, A. R. Hyde, J. B. Kelly, Joseph Lehner, Stanislaw Leja, D. C. Lewis, Viktors Linis, A. E. Livingston, D. C. B. Marsh, D. J. Newman, S. Parameswaran, George Piranian, H. W. E. Schwerdtfeger, Michael Skalskyj, O. E. Stanaitis, G. R. Trimble, Jr., Chih-yi Wang, Morgan Ward, J. V. Whittaker, and the Proposer.

* See Polya–Szegö, Aufgaben und Lehrsätze aus der Analysis, vol. 1, p. 53, Problem 74.

Topological Space in Which Every Permutation Is Continuous

4548 [1953, 482]. *Proposed by Joel Pitcairn and Robert Ellis, University of Pennsylvania*

Let X be a topological space with the property that every one-to-one transformation of X onto itself is continuous. Characterize the open subsets of X.

Solution by H. E. Vaughan, University of Illinois. It is slightly more convenient to characterize the closed subsets. Using "condensed" to describe a space which has no closed, non-empty, proper subsets and "discrete" to describe one each of whose subsets is closed, one has the following

THEOREM. *A space X has the property that every one-to-one transformation of X onto itself is continuous if and only if X is either condensed or discrete or there exists a transfinite cardinal $n \leqq \overline{X}$ such that a proper subset F of X is closed if and only if $\overline{F} < n$.* ($\overline{F}$ *is* the cardinal number of the set F.)

Proof: The sufficiency is obvious. Suppose, then, that every one-to-one transformation of X onto itself is continuous (and hence a homeomorphism) and that X is neither condensed nor discrete. Let F be a closed non-empty proper subset of X, p a point of F, and q a point of $X-F$. Then, clearly, $(F-\{p\})+\{q\}$ is also closed. Hence every subset of F is, as an intersection of F and such sets, also closed. In particular X is a T_1-space and, since it is not discrete, is infinite. Furthermore, X is not the join of two closed proper subsets: If it were, then the two sets could, by the preceding result, be chosen disjoint. Now if p is any point of X and F is that one of the two sets to which p belongs, while q is a point of $X-F$, then $((F-\{p\})+\{q\})+(X-F)=X-\{p\}$ is closed. But this contradicts the assumption that X is not discrete.

It now follows that no closed proper subset of X has the same cardinal number as X. For if F were such a set then one would have $\overline{X-F} \leqq \overline{X} = \overline{F}$ and there would be a one-to-one transformation of X on itself mapping $X-F$ in F. (If $\overline{X-F} = \overline{F}$, merely interchange the two sets. If $\overline{X-F} < \overline{F}$, then map $X-F$ on a subset F_1 of F. Since $\overline{F}_1 < \overline{X}$ and X is infinite, $\overline{X-F}_1 = \overline{X} = \overline{F}$ and the transformation can be completed by mapping F on $X-F_1$.) Since every subset of F is closed and the transformation is continuous, $X-F$ is closed. But X is not the join of two closed proper subsets. Furthermore, if F is a closed proper subset of X then so is every subset of X which has the same cardinal number. For if F_1 is such a set, then, since $\overline{F} < \overline{X}$, $\overline{X-F} = \overline{X} = \overline{X-F_1}$ and there exists a one-to-one transformation mapping F_1 on F and $X-F_1$ on $X-F$. Since this transformation is continuous, F_1 is closed.

It is now clearly sufficient to take n to be the least cardinal number greater than that of each closed proper subset of X.

It would be of interest to find a solution which does not, like the above, depend on the axiom of choice.

Also solved by G. U. Brauer, Helen F. Cullen, David Ellis and Robert Bagley, Azriel Rosenfeld, J. J. Schäffer, W. R. Scott, G. H. M. Thomas, and the Proposers.

The Four Feuerbach Tangents

4549 [1953, 482]. *Proposed by Richard Oblâth, Budapest, Hungary*

The Gauss-Newton line of the complete quadrilateral formed by the four Feuerbach tangents of a triangle is the Euler line of the triangle.

Solution by R. Goormaghtigh, Bruges, Belgium. The property has been given before by W. Gallatly, *Mathesis*, 1908, p. 33. It may be derived immediately from the fact that the Feuerbach tangents are the reciprocal lines to those joining the centroid G to the in- and excenters; hence, the considered tangents are common to the nine-point circle and the inner Steiner ellipse, and the Newton line of the quadrilateral formed by these tangents, being the locus of the centers of the conics tangent to them, is the Euler line which passes through the nine-point center and through G.

Generalization. The present writer has proved (*Mathesis*, 1922, p. 164) that if P is one of the points where a circumdiameter meets the *seventeen-point cubic*,* the orthopole of that diameter is the projection of P on the reciprocal line to PG. Hence:

If P_1, P_2, P_3, P_4 are any four points on the seventeen-point cubic, the perpendiculars erected at the orthopoles of the circumdiameters passing through P_1, P_2, P_3, P_4 on the lines joining these orthopoles to P_1, P_2, P_3, P_4, respectively, form a quadrilateral, the Newton line of which passes through the centroid.

When the points P_1, P_2, P_3, P_4 are the in- and excenters, the considered perpendiculars are tangent to the nine-point circle, and the Newton line passes then also through the center of that circle.

Consecutive Cubes Whose Difference Is a Triangular Number

4550 [1953, 482]. *Proposed by P. L. Chessin, Cooper Union, New York City*

I. The smallest consecutive cubes whose difference is a triangular number are 6^3 and 5^3. What are the next two pairs?

II. Under the assumption in I, show that this difference is equal to the difference of two squares of the form $(9a+1)^2-9a^2$.

To Victor Thébault this problem is humbly dedicated.

Solution by R. Venkatachalam Iyer, Trivandrum, India. The hypothesis gives

$$(N+1)^3 - N^3 = x(x+1)/2$$

which reduces to the Pellian

$$6q^2 + 3 = p^2$$

with $p=2x+1$ and $q=2N+1$. Positive integral solutions are given from $p_i+q_i\sqrt{6}=(3+\sqrt{6})(5+2\sqrt{6})^i$, and both p and q satisfy the recursion formula $u_i=10u_{i-1}-u_{i-2}$. The consecutive cubes, N^3 and $(N+1)^3$ are now easily found

* Locus of the points such that the joins of these points to their counter-points pass through the centroid.

to be 5^3 and 6^3, 54^3 and 55^3, 539^3 and 540^3, $\cdots$.

From $p_1=3$ and $p_2=27\equiv 3 \pmod{24}$ and the recursion formula, it follows that every $p_i\equiv 3 \pmod{24}$. Then $p=24a+3$ implies $x=12a+1$, whence

$$(N+1)^3 - N^3 = (12a+1)(12a+2)/2 = (9a+1)^3 - 9a^3.$$

Also solved by Norman Anning, Leon Bankoff, W. J. Blundon, W. B. Carver, M. A. Kirchberg, D. C. B. Marsh, Eric Michalup, S. Parameswaran, M. J. Pascual, L. A. Ringenberg, R. E. Shafer, G. W. Walker, Chih-yi Wang and the Proposer, many of whom referred to the analogous problems E 702 [1946, 464] and 4299 [1950, 189]. and also a note by V. Thébault [1949, 174].

A Limit Involving the Gamma Function

4551 [1953, 482]. *Proposed by K.-F. Moppert, Basle, Switzerland*

For $k=0, 1, 2, \cdots$, prove

$$\lim_{n\to\infty} n^{\alpha-k} \sum_{\nu=0}^{n-1} (-1)^\nu \gamma^k \binom{\alpha}{\gamma} = \frac{1}{(k-\alpha)\Gamma(-\alpha)}.$$

Remark: for $\alpha=-1$ this gives a well known asymptotic value for the sum of the kth powers.

Solution by Joseph Lehner, Los Alamos Scientific Laboratory. We first prove the formula for $k=0$, $\alpha\neq 0, 1, 2, \cdots$. Since

$$\binom{\alpha}{\nu} = \binom{\alpha-1}{\nu} + \binom{\alpha-1}{\nu-1}, \qquad \nu \geqq 1,$$

and $\binom{\alpha}{0} = \binom{\alpha-1}{0} = 1$, we get

$$\sum_{\nu=0}^{n-1} (-1)^\nu \binom{\alpha}{\nu} = \sum_{\nu=0}^{n-1} (-1)^\nu \binom{\alpha-1}{\nu} + \sum_{\nu=1}^{n-1} (-1)^\nu \binom{\alpha-1}{\nu-1}$$

$$= (-1)^n \binom{\alpha-1}{n-1} = \binom{-\alpha+n-1}{n-1}$$

$$\sim \frac{n^{-\alpha}}{\Gamma(-\alpha+1)}, \qquad n\to\infty,$$

by Stirling's formula.

Now suppose the proposed formula proved for $0\leqq k<j$ and $\alpha\neq 0, 1, 2, \cdots$. We use

$$\binom{\alpha-1}{\nu} = \binom{\alpha}{\nu}\left[1-\frac{\nu}{\alpha}\right], \qquad \nu = 0, 1, 2, \cdots,$$

and obtain

$$\sum_{\nu=0}^{n-1} (-1)^\nu \nu^{j-1} \binom{\alpha-1}{\nu} = \sum_{\nu=0}^{n-1} (-1)^\nu \nu^{j-1} \binom{\alpha}{\nu} - \frac{1}{\alpha} \sum_{\nu=0}^{n-1} (-1)^\nu \nu^j \binom{\alpha}{\nu}.$$

When α is not equal to a non-negative integer, this is true also of $\alpha-1$. We apply the induction hypothesis, getting

$$\lim_{n\to\infty} n^{\alpha-j}\sum_{\nu=0}^{n-1}(-1)^\nu\nu^j\binom{\alpha}{\nu} = \alpha\lim_{n\to\infty} n^{\alpha-j}\sum_{\nu=0}^{n-1}(-1)^\nu\nu^{j-1}\binom{\alpha}{\nu}$$

$$-\alpha\lim_{n\to\infty} n^{\alpha-1-(j-1)}\sum_{\nu=0}^{n-1}(-1)^\nu\nu^{j-1}\binom{\alpha-1}{\nu}$$

$$= 0-\frac{\alpha}{(j-\alpha)\Gamma(-\alpha+1)} = \frac{1}{(j-\alpha)\Gamma(-\alpha)},$$

as required.

Both members of the formula are continuous functions of α for all α and so it still holds for $\alpha=0, 1, 2, \cdots$. The right member then has the value

$$\frac{d}{dx}\left(\frac{1}{\Gamma(x)}\right)_{x=-k}$$

for $\alpha=k$, and 0 for $\alpha\neq k$.

Also solved by H. W. Gould, O. E. Stanaitis, and Chih-yi Wang.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

First Course in Abstract Algebra. By R. E. Johnson. New York, Prentice-Hall, Inc., 1953. 8+257 pages. $5.50.

This book is an introductory text in abstract algebra, intended as a year course for upper division students. It is not a survey of modern algebra, but is prepared as a working text on basic material, in which the hand and mind of the student are carefully guided, by easy stages and frequent illustrative examples, from concrete to abstract.

Starting with an introduction to mappings and relations, the elementary number systems, through the reals, are developed from the Peano axioms. Various categories of rings—integral domains, fields, polynomial rings—are studied in some detail, while the concept ring *per se* and the concept of ideal are only very lightly touched on at the conclusion of the book. Abstract groups and the symmetric group are treated in some twenty-five pages, ending with the mapping theorem. Linear theory—matrices, linear equations and determinants—are presented in modern form (about sixty pages). The book concludes with a very

short discussion of Boolean algebras.

In the opinion of the reviewer this is a significant contribution to the literature of this intermediate level. The arguments are clear and the student should be eased into the spirit of modern algebra without being frightened by too much generality or pathology. An adequate supply of exercises is offered, most of which are well within the range of the average student.

While a selection of material is of course required, it nevertheless seems unfortunate that the author thought it desirable to completely omit such conventional material as characteristic value theory and quadratic forms. Also a somewhat expanded treatment of groups, with a few applications, would have been welcome.

A. L. FOSTER
University of California

Principles of Numerical Analysis. By A. S. Householder. New York, McGraw-Hill, Inc., 1953. x+274 pages. $6.00.

With the recent increase in the application of science to practical problems and the attendant requirement for numerical answers has come a demand for computational procedures which are more efficient and suited to modern computing machinery. This has led to a resurgence of interest in numerical analysis and the development of new computational techniques with the result that many methods applicable to a wide range of special cases are now known. Many are new, others are modifications and refinements of older methods, some of which had long been forgotten. The results of these researches of the past ten years are scattered through the literature, both formal and informal, much of it not readily available nor widely known.

In *Principles of Numerical Analysis* A. S. Householder has assembled many of these results in a form conveniently available to the computer. This is a real service which will be welcomed by all who are interested in the art of computation. In spite of the author's statement in the preface that this text can be read by those who have had a course in calculus, this would seem to be a book for readers with some mathematical maturity and particularly with some knowledge of the theory of matrices and vector spaces. The treatment of these topics in the text could be improved and, instead of trying to make the book self-contained, it might have been better to have omitted the theory given there and have referred the reader to one of the standard works on the subject. The author's approach throughout the book is theoretical which should be welcomed by those mathematicians who have tried to learn numerical analysis by following detailed computations. Theories are given in their most general form, often containing as special cases older standard methods of computation, as the author points out. The author has limited himself for the most part to algebraic problems and has not treated at all the large field of differential and integral equations. This subject, however, is better known and has already been covered in several excellent books.

This reviewer was glad to see that the subject of errors has not been neglected. The book opens with a general discussion of the nature and source of errors arising in computation and later when treating special methods error estimates are frequently given. It would have been desirable to have had more of this, for an estimate of the error involved is an important part of any calculation

A large part of the book is devoted to computational problems involving matrices. Direct and iterative methods for solving systems of linear equations and finding the characteristic roots and vectors of matrices are treated. For many of the methods operational counts are given. These are the approximate number of multiplications, divisions and recordings required in the computation which, when using digital computers, is useful in estimating time requirements. The remainder of the book is devoted to non-linear equations, interpolation, orthogonal functions and numerical differentiation and integration. The book closes with a brief description of the Monte Carlo method. It is too bad there is not more on this important new computational technique. There is an extensive bibliography.

Unfortunately, the author has allowed an occasional inaccurate statement to mar the text. He frequently uses "complex" when he means "non-real." On page 122 occurs the statement " . . . if the equation is real, then the complex roots occur in conjugate pairs, . . . ," which is not true as it stands. The author often uses algebraic equation when he means polynomial equation. The expressions on page 138 for $u(x, y)$ and $v(x, y)$ are true if $f(z)$ is a polynomial with real coefficients. These are minor points which should cause the experienced reader no trouble, but they do detract from the over-all appearance of the book.

This should be a useful book, both as a reference for the experienced computer and as a text for one who wishes to learn the subject.

W. H. DURFEE
Operations Research Office

1. *Trigonometry*. By J. F. Randolph. New York, The Macmillan Company, 1953. 10+186+34 pages. $3.00.
2. *Plane Trigonometry*. By P. R. Rider. New York, The Macmillan Company, 1953. 8+152+28 pages. $3.00.
3. *Plane Trigonometry*, Second Edition. By A. W. Weeks and H. G. Funkhouser. New York, D. Van Nostrand and Company, Inc., 1953. 8+193+42 pages. $2.68 (without tables), $2.88 (with tables).

A successful editor once said that a good text should have at least one innovation. These three texts all qualify in this respect.

In their careful attention to form and presentation and teaching devices the authors are doing something about the difficult problem of teaching trigonometry. Each of these three texts meets the needs of the course for which it was designed.

If the present tendency to use smaller and still smaller type continues the student will soon need a high power magnifier. Since the war is terminated surely paper is not so scarce or expensive.

This reviewer is not commenting on the tables except to state that if the tables in each text are not considered adequate to meet the needs of a particular institution, separate tables are available.

Randolph seems to have introduced more innovations than the others. The distance formula is made basic in the subject yet his approach may cause the reader to overlook its importance. The definition of the co-function on page 34 is excellent. The use of projections in writing coordinates is very good. But has the author overlooked many of the advantages of the use of the projection theorems?

An important teaching aid is the use of the identity sign $\equiv$ instead of the equal sign $=$ when working with identities. The introduction of the scale factor a in the multiple angle formulas gives generality in a sense but may cause confusion in the mind of the student. On page 82 the introduction of the idea that if the product of two numbers is zero then at least one of the numbers is zero emphasizes an important concept that is frequently overlooked.

It will be interesting to note the reaction of educators to the introduction of complex numbers as the algebra of ordered pairs. From the advanced viewpoint it has much to recommend it. From the teaching approach is there already so much material (new to the student) in the course, that additional ideas serve only to confuse? This is a text which the experienced teacher will need to study carefully before each class exercise. It has a wonderful new approach and it seems especially good for the gifted student.

The Rider text is well written. The approach to logarithms and approximate numbers is excellent. The stress on the importance of an orderly arrangement of the work in numerical trigonometry is almost nil, although he uses a good form in the examples. The derivation of the law of tangents by geometric methods is good. The approach to Heron's formula is interesting. The treatment of vectors is brief. In view of its applications, has it been given sufficient attention? The material is condensed and the student should grasp the ideas readily.

The text by Weeks and Funkhouser is an excellent standard text with a large number of good problems. The check formula on page 55 by the use of projections is different and interesting. The development of $\sin(A+B)$ by areas is an easy way for the student to grasp the relationship. The chapter on applications is good but some instructors may object to the emphasis given to problems connected with the military.

The authors include a little spherical trigonometry in the applications to navigation.

These three texts are all excellent and each one contains excellent teaching aids.

FRED ROBERTSON
Iowa State College

NEW BOOKS RECEIVED

Differential and Integral Calculus. By C. E. Love and E. D. Rainville. New York, The Macmillan Company, 1954. xiv+526 pages.

Fundamentals of College Mathematics. By J. C. Brixey and R. V. Andree. New York, Henry Holt and Company, 1954. xiv+609 pages. $5.90.

The Method of Trigonometrical Sums In the Theory of Numbers. By I. M. Vinogradov. New York, Interscience Publishers, Inc., 1954. x+180 pages. $5.00.

Gas Dynamics of Thin Bodies. By F. I. Frankl and E. A. Karpovich (Translated by M. D. Friedman). New York, Interscience Publishers, Inc., 1954. viii+175 pages. $5.75.

Eingefrangenes Unendilich Bekenntnis Zur Geschichte Der Mathematik. By Franz Von Krbek. Leipzig, Geest and Portig K. G., 1954. iv+332 pages. $5.25.

Analog Methods in Computation and Simulation. By W. W. Soroka. New York, McGraw-Hill Book Company, Inc., 1954. xii+390 pages. $7.50.

The Kinematics of Vorticity. By C. Truesdell. Bloomington, Indiana, Indiana University Press, 1954. xvii+232 pages. $6.00.

Wave Motion and Vibration Theory. Volume V. Proceedings of the Fifth Symposium in Applied Mathematics of the American Mathematical Society. Editor: Albert E. Heins. New York, McGraw-Hill Book Company, Inc., 1954. v+169 pages. $7.00.

Mathematics for the Secondary School. By W. D. Reeve. New York, Henry Holt and Company, 1954. xii+547 pages. $5.95.

Theoretical Physics. By F. W. Constant. Cambridge, Addison-Wesley Publishing Company, Inc., 1954. xiv+281 pages. $6.50.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

NUMERICAL ANALYSIS RESEARCH, UCLA

Numerical Analysis Research, University of California at Los Angeles, is continuing much of the research program of the Institute for Numerical Analysis (INA) of the National Bureau of Standards; INA was disbanded on June 30, 1954.

The new organization, administered by the UCLA Mathematics Department, will receive support from the Office of Naval Research and the Office of Ordnance Research. Its primary mission includes pertinent fundamental research in mathematics and science and research in the application of computers to problems occurring in science and other applied fields. It will be aided in at-

taining these objectives by the use of the National Bureau of Standards Western Automatic Computer (SWAC) and other equipment which has been loaned to UCLA. In addition, the new organization has the use of INA's library. Most of the research staff of INA has joined the new organization.

The organization offers a training program in the efficient application of high-speed digital computers. A number of graduate assistantships offered in co-operation with various university departments are available each year. Seminars and formal and informal courses in numerical analysis are conducted.

Correspondence should be addressed to: Numerical Analysis Research, University of California, 405 Hilgard Avenue, Los Angeles 24, California.

PRELIMINARY ACTUARIAL EXAMINATIONS PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of Part 2 of the 1954 Preliminary Actuarial Examination are as follows:

First Prize of $200

Monsky, Paul	Swarthmore College

Additional Prizes of $100

Bowers, N. L.	Yale University
Croteau, Robert	University of Montreal
Driscoll, F. T.	Yale University
Fike, C. T.	University of the South
Freeman, D. N.	Yale University
Huff, R. W.	College of Wooster
Reinken, D. L.	Princeton University
Shapland, Robert	Drake University
Strang, W. G.	Massachusetts Institute of Technology

The Society of Actuaries has authorized a similar set of nine prizes for the 1955 examinations on Part 2.

The Preliminary Actuarial Examination consists of the following three parts:

Part 1. *Language Aptitude Test.*
(Reading comprehension, meaning of words and word relationships, antonyms, and verbal reasoning.)

Part 2. *General Mathematics Examination.*
(Algebra, trigonometry, coordinate geometry, differential and integral calculus.)

Part 3. *Special Mathematics Examination.*
(Finite differences, probability and statistics.)

The 1955 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service and will be administered by the Society of Actuaries at centers throughout the United States and Canada on May 11, 1955 (tentative

date). The closing date for applications is March 15, 1955.

Detailed information concerning the Examinations can be obtained from: The Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

The Society for Industrial and Applied Mathematics will hold its first national meeting in conjunction with the annual meetings of the American Mathematical Society, the Mathematical Association of America, and the Association for Symbolic Logic at the University of Pittsburgh on December 27–29. The following addresses will be presented to an evening meeting: "The History of a Problem," by Dr. Brockway McMillan, Bell Telephone Laboratories; "The Control of Industrial Operations," by Prof. Herbert A. Simon, Carnegie Institute of Technology; "Probability Theory in Liability and Property Insurance," by Mr. C. W. Crouse, Actuary, Preslan and Company. Further information can be obtained from H. W. Kuhn, Dalton Hall, Bryn Mawr College, Bryn Mawr, Pa.

PERSONAL ITEMS

Professor H. S. Pollard of Miami University was appointed to represent the Association at the inauguration of President H. B. Young, Western College for Women, on October 9, 1954.

Professor T. J. Higgins of the University of Wisconsin has been presented with the George Westinghouse award of $1,000 for outstanding teaching by the American Society for Engineering Education.

Dr. W. A. Shewhart has been given the first Honorary Professorship in Statistical Quality Control by Rutgers University.

Assistant Professor J. V. Talacko of Marquette University has received a Ford Foundation Fellowship and plans to spend most of the year at the Statistical Laboratory, University of California.

The Educational Testing Service announces the following awards: Mr. R. F. Boldt, graduate student at Princeton University, and Mr. D. P. Estavan, graduate student at Stanford University, have received fellowship awards for graduate study in psychology at Princeton University; Mr. J. A. Keats, member of the staff of the Australian Council for Educational Research in Melbourne, Australia, has been awarded the Visiting Psychometric Fellowship for the second year; Mr. Norman Cliff, Mr. Bertram Karon, Mr. Anton Morton, and Mr. Robert Sadacca have been re-appointed as Psychometric Fellows.

Mr. M. L. Anthony, former senior research engineer at the Armour Research Foundation, Chicago, Illinois, is a research engineer at American Machine and Foundry Company, Chicago.

Dr. R. W. Bagley of the University of Florida has accepted a position as an assistant professor at the University of Kentucky.

Mr. E. H. Batho, previously a teaching assistant at the University of Wisconsin, has been appointed to an instructorship at the University of Rochester.

Mr. R. E. Bayles, a graduate student of Harvard University, is an actuarial assistant for the John Hancock Mutual Life Insurance Company, Boston, Massachusetts.

Assistant Professor Helen P. Beard of Newcomb College has returned from her leave of absence at the Statistical Laboratory, University of California.

Miss Barbara J. Beechler, formerly an instructor at Smith College, is now at the State University of Iowa as a part-time instructor.

Professor L. M. Blumenthal of the University of Missouri will be on leave during 1954–55 to serve as Fulbright Lecturer at the University of Leiden, The Netherlands.

Dr. R. N. Bradt, previously a graduate assistant at Stanford University, has been appointed to an assistant professorship at the University of Kansas.

Mr. A. R. Brown, Jr., recently a mathematician at Aberdeen Proving Ground, Maryland, has been appointed Associate Professor at Drury College.

Associate Professor F. J. H. Burkett of Union College has been promoted to a professorship.

Assistant Professor V. B. Caris of Ohio State University has retired.

Mr. R. C. Courter, formerly a student at Columbia University, is now a mathematician at the Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Mr. G. L. Crumley, recently a graduate student at the University of Kansas, is now an instructor at Coffeyville College.

Mr. R. A. Dean, previously a lieutenant in the U. S. Navy, is a fellow at the California Institute of Technology.

Assistant Professor R. D. Depew of Florence State Teachers College, Alabama, has been promoted to an associate professorship.

Associate Professor R. J. Dunholter of the University of Cincinnati has been promoted to a professorship.

Reverend A. J. Eiardi, formerly chairman of the Mathematics Department of Boston College, has accepted a position as an associate professor at Fairfield University.

Mrs. Cecile S. Feder, instructor at Hunter College, has been appointed Registrar and Assistant Professor of Mathematics at Stern College for Women, Yeshiva University.

Mr. G. F. Feeman, graduate assistant at Lehigh University, has been appointed Instructor in Mathematics and Physics at Muhlenberg College.

Mr. H. H. Fox, previously a research physicist at Mound Laboratories, Miamisburg, Ohio, has accepted a position as a mathematician at the Applied Physics Laboratory, Johns Hopkins University.

Mr. R. A. Gambill, a former member of the Federal Public Housing Administration, Indianapolis, Indiana, is engaged as a mathematician at the U. S. Naval Ordnance Laboratory, Indianapolis.

Assistant Professor S. T. Gormsen of the University of Florida has been appointed to a professorship at Rollins College.

Mr. J. C. Gould, recently a student at Wabash College, is with the United States Army as a supply records specialist.

Professor Simon Green of Philander Smith College has accepted a position as an assistant professor at the University of Tulsa.

Mr. R. L. Gulley, formerly a mathematician with the Bureau of Ships, Navy Department, Washington, D. C., is a mathematician with the Naval Electronics Laboratories, Point Loma, California.

Dr. Henry Helson of Yale University has been promoted to an assistant professorship.

Associate Professor Carl Holtom of the U. S. Air Force Institute of Technology, Wright-Patterson Air Force Base, Dayton, Ohio, has resigned to become a staff member of the University of California's Los Alamos Scientific Laboratory.

Mrs. Betty W. Holz, who was an engineer with Melpar Incorporated, Alexandria, Virginia, has accepted a position as an operations research analyst with Johns Hopkins University.

Professor S. B. Jackson of the University of Maryland has succeeded Professor M. H. Martin as Head of the Mathematics Department.

Professor E. F. W. Jones of Rollins College retired in June, 1954.

Professor Chosaburo Kato, formerly chairman of the Mathematics Department of Denison University, has been promoted to the newly created senior professorship.

Mr. M. A. Kirchberg, former student at Eastern Illinois State College, has accepted a position as teacher at Hopkins Township School, Hopkins, Michigan.

Mr. Charles Kurland, a project engineer at the Chevrolet Aviation Engine Company, Buffalo, New York, is now a test engineer for the Chevrolet Engineering Laboratory, Detroit, Michigan.

Mr. Milton Lees, a former student at the University of California, has been promoted to a research assistantship.

Professor S. L. Levy of Brown University has been appointed Manager of the Applied Physics Division of the Midwest Research Institute.

Professor M. H. Martin of the University of Maryland has been appointed Director of the University's Institute for Fluid Dynamics and Applied Mathematics.

Mr. K. A. McGown, former instructor at Lafayette College, is teaching at Boonton High School, Boonton, New Jersey.

Mr. Michael Montalbano, previously employed as Chief of the Programming Section of the Computer Control Company, Port Mugu, California, is now a mathematician at the Kaiser Steel Corporation, Fontana, California.

Reverend T. F. Mulcrone has been transferred from Spring Hill College to St. Charles College.

Mr. R. C. Nickerson, recently a student at Brown University, has a position as engineer at the General Electric Company, Physics Training Program, Utica, New York.

Mr. A. H. Payne, formerly a mathematician at the Ballistics Research Laboratories, Aberdeen Proving Ground, Maryland, is an analyst for the Raytheon Manufacturing Company, Waltham, Massachusetts.

Mr. W. J. Pervin, previously a teaching assistant at Carnegie Institute of Technology, is a senior scientist at Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania.

Dr. S. R. Peterson of Union College has been promoted to the position of Assistant Professor of Philosophy.

Assistant Professor T. J. Pignani of Loyola University is at the University of North Carolina for the year 1954–55.

Mr. E. J. Polak, senior project engineer at the Curtiss-Wright Corporation, Carlstadt, New Jersey, has been appointed to an assistant professorship at Bucknell University.

Associate Professor S. E. Rauch has been promoted to a professorship at the University of California, Santa Barbara.

Mr. Saul Rosen, associate research engineer at Burroughs Adding Machine Company, Philadelphia, Pennsylvania, has been appointed to an associate professorship at Wayne University and is a member of the staff of the Computation Laboratory.

Mr. A. I. Rosenfeld, a graduate student at Columbia University, has a position as a junior physicist at the Freed Electronics and Controls Corporation, New York City.

Professor W. E. Roth of the University of Tulsa has resigned from this position and retired from teaching.

Miss Miriam J. Russell, previously an instructor at the University of Arizona, is now a teacher at Wakefield High School, Arlington, Virginia.

Associate Professor W. A. Rutledge of Alabama Polytechnic Institute has accepted a position as an assistant professor at the University of Tulsa.

Mr. W. B. Stovall, Jr., formerly Chief of the Vital Statistics Section of the State Board of Health, Jacksonville, Florida, has accepted a position as instructor at the University of Florida.

Associate Professor W. B. Temple of Louisiana Polytechnic Institute has been promoted to a professorship.

Dr. L. E. Ward, Jr., of the University of Nevada has been appointed to a professorship at the University of Utah.

Mr. W. G. Younkin, formerly a student at the University of California, Los Angeles, is a physicist at the U. S. Naval Ordnance Test Station, China Lake, California.

Emeritus Professor E. W. Pehrson of the University of Utah died on October 11, 1953. He was a charter member of the Association.

Emeritus Dean R. P. Stephens of the University of Georgia died on June 1, 1954. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-FIFTH SUMMER MEETING OF THE ASSOCIATION

The thirty-fifth summer meeting of the Mathematical Association of America was held at the University of Wyoming, Laramie, Wyoming, on Monday and Tuesday, August 30 and 31, 1954, in conjunction with the summer meetings of the American Mathematical Society and the Canadian Mathematical Congress. About four hundred and fifty adults were registered including the following two hundred and thirty-five members of the Association:

L. V. Ahlfors, C. B. Allendoerfer, J. M. Anderson, R. D. Anderson, R. V. Andree, H. A. Antosiewicz, T. M. Apostol, Nachman Aronszajn, W. F. Atchison, D. F. Atkins, R. W. Ball, C. F. Barr, M. A. Basoco, S. Louise Beasley, E. M. Beesley, E. G. Begle, B. C. Bellamy, J. S. Bendat, A. I. Benson, Arthur Bernhart, R. H. Bing, R. G. Blake, G. U. Brauer, W. E. Briggs, R. W. Brink, Leonard Bristow, J. R. Britton, Myrtle C. Brown, R. G. Buschman, R. K. Butz, J. E. Byrne, R. L. Calvert, F. M. Carpenter, R. E. Carr, Harold Chatland, R. V. Churchill, S. D. Conte, T. F. Cope, K. L. Cooke, W. R. Cowell, H. M. Cox, R. B. Crouch, A. B. Cunningham, J. H. Curtiss, P. H. Daus, W. M. Davis, Jim Douglas, Jr., T. C. Doyle, D. M. Dribin, W. L. Duren, Jr., J. C. Eaves, G. M. Ewing, H. J. Fletcher, J. S. Frame, W. B. Fulks, M. G. Gaba, J. W. Gaddum, R. A. Gambill, A. E. Gault, H. M. Gehman, Michael Goldberg, Michael Golomb, S. H. Gould, H. T. Guard, R. R. Gutzman, W. T. Guy, Jr., Marian S. Gysland, Edwin Halfar, P. C. Hammer, J. R. Hanna, W. L. Hart, C. L. Hassell, Jr., F. C. Hatfield, Nola L. A. Haynes, Leota C. Hayward, T. J. Head, I. L. Hebel, E. R. Heineman, Melvin Henriksen, Anna S. Henriques, J. G. Herriot, I. N. Herstein, Fritz Herzog, J. F. Heyda, J. J. L. Hinrichsen, S. T. Hu, Ralph Hull, J. A. Hummel, M. Gweneth Humphreys, N. C. Hunsaker, J. W. Hurst, W. R. Hutcherson, C. A. Hutchinson, Jane C. Ingersoll, L. K. Jackson, R. D. James, C. A. Johnson, L. W. Johnson, Ernest Johnston, L. S. Johnston, D. H. Jones, F. B. Jones, P. S. Jones, Dora E. Kearney, M. W. Keller, J. B. Kelly, D. E. Kibbey, E. C. Kiefer, V. L. Klee, Jr., J. C. Knipp, F. W. Kokomoor, R. B. Kriegh, Ruth G. Lane, C. E. Langenhop, Leo Lapidus, E. H. Larguier, W. I. Layton, J. R. Lee, D. H. Lehmer, R. B. Leipnik, W. J. LeVeque, Gene Levy, F. A. Lewis, C. H. Lindahl, W. D. Lindstrom, Charles Loewner, A. J. Lohwater, R. G. Long, A. T. Lonseth, L. H. Loomis, Lee Lorch, M. L. Madison, Morris Marden, Margaret E. Mauch, B. H. McCandless, Garner McCrossen, W. C. McDaniel, A. W. McGaughey, S. W. McInnis, J. H. McKay, E. B. McLeod, Jr., D. F. Mela, A. S. Merrill, B. C. Meyer, R. J. Michel, R. J. Mihalek, E. B. Miller, C. B. Morrey, Jr., E. D. Mouzon, Jr., M. E. Mullings, C. H. Murphy, Jr., W. H. Myers, Greta Neubauer, T. A. Newton, C. O. Oakley, Morris Ostrofsky, T. G. Ostrom, L. J. Paige, B. J. Pettis, C. G. Phipps, George Piranian, Everett Pitcher, G. B. Price, M. H. Protter, W. T. Puckett, Jr., Henry Rainbow, Gordon Raisbeck, J. F. Randolph, O. M. Rasmussen, G. E. Raynor, O. H. Rechard, O. W. Rechard, A. W. Recht, P. V. Reichelderfer, Haim Reingold, H. B. Ribeiro, D. E. Richmond, F. A. Rickey, J. D. Riley, E. K. Ritter, Calvin A. Rogers, G. F. Rose, Mary E. Rudin, R. G. Sanger, Robert Schatten, O. F. G. Schilling, Nathan Schwid, W. R. Scott, George Seifert, A. L. Shields, Sister Mary Corona, Sister Mary Felice, Abe Sklar, M. F. Smiley, A. J. Smith, G. W. Smith, S. R. Smith, W. N. Smith, L. C. Snively, Andrew Sobczyk, J. McD. Staley, R. D. Stalley, D. W. Starr, P. O. Steen, B. M. Stewart, Ruth W. Stokes, E. B. Stouffer, D. R. Sudborough, J. G. Sutton, P. M. Swingle, J. M. Thomas, C. J. Thorne, R. M. Thrall, W. J. Thron, Wilmont Toalson, C. B. Tompkins, Leonard Tornheim, C. W. Trigg, A. W. Tucker, W. R. Utz, Jr., V. J. Varineau, R. W. Veatch, G. L. Walker, J. A. Ward, L. A. Ware, M. S. Webster, J. G. Wendel, A. L. Whiteman, L. R. Wilcox, W. L. Williams, G. M. Wing, R. M. Winger, Oswald Wyler, C. R. Wylie, Jr., Max Wyman.

Sessions of the Association were held on Monday morning and afternoon and on Tuesday morning in the Education Auditorium of the University of Wyoming, with Professors C. F. Barr, R. D. James, and J. S. Frame presiding. The third series of Earle Raymond Hedrick Lectures was delivered by Professor L. H. Loomis of Harvard University. The Program Committee for the meeting consisted of Marshall Hall, Jr., Chairman; Harold Chatland, B. E. Meserve, and G. deB. Robinson.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Convex Sets," Lecture I, by Professor L. H. Loomis, Harvard University.

Symposium on Educational Aspects of Computers

"The Educational Impact of the Illiac," by Professor W. F. Atchison, University of Illinois.

"Effects of Large Digital Computers on Numerical Analysis Curricula," by Professor C. B. Tompkins, University of California at Los Angeles.

"A Machine's-eye View of Numerical Analysis," by Professor D. H. Lehmer, University of California.

SECOND SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Convex Sets," Lecture II, by Professor L. H. Loomis, Harvard University.

Joint Session with the Canadian Mathematical Congress

"Some Geometrical Applications of Taylor's Formula," by Professor Peter Scherk, University of Saskatchewan.

"Asymptotic Expansions," by Professor Max Wyman, University of Alberta.

"Systems of Congruences," by Professor B. M. Stewart, Michigan State College.

THIRD SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Convex Sets," Lecture III, by Professor L. H. Loomis, Harvard University.

"The Preparation of College Mathematics Teachers," by Professor P. V. Reichelderfer, Ohio State University.

"Integral Transformations and Differential Equations," by Professor R. V. Churchill, University of Michigan.

"Metrics and Matrices," by Professor A. T. Lonseth, Oregon State College.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the Education Lounge of the University of Wyoming. Twenty-one of the forty

members of the Board were present. Among the more important items of business transacted were the following:

Professor E. N. Oberg of the State University of Iowa was elected to serve as Governor from the Iowa Section for a term expiring in June 1956 to succeed the late Professor D. L. Holl.

Professor Mark Kac of Cornell University was invited to deliver the fourth series of Hedrick Lectures at the 1955 Summer Meeting.

Announcement was made of the following grants received by the Association during the current year: a grant of $15,000 from the National Science Foundation for the support of the Program of Visiting Lecturers; a grant of $2,500 from the Social Science Research Council for the work of the Committee on the Undergraduate Program; a grant of $1,000 from the National Research Council to assist with the publication of Slaught Paper Number 3 which contains the "Proceedings of the Symposium on Special Topics in Applied Mathematics." In each case, the Association will supplement from its own funds the amounts granted by these other organizations.

It was announced that the membership of the Association was 5,615 on August 20.

MEETING OF SECTION OFFICERS

A meeting of Officers of the Sections of the Association was held on Tuesday evening in the Education Lounge. Thirty-four persons were present representing twenty-three of the twenty-five Sections of the Association. The following topics were discussed: publication of accounts of Section Meetings in the MONTHLY, the Program of Visiting Lecturers, the Committee on Employment Opportunities, programs of Section Meetings, finances of the Sections, Section activities, such as contests and lecturers, and reports of special committees of certain Sections. A report was distributed from the Illinois Section Committee on Strengthening the Teaching of Mathematics in Secondary Schools.

MEETINGS OF OTHER ORGANIZATIONS

Sessions of the American Mathematical Society began on Tuesday afternoon and continued through Friday morning. Invited addresses were given by Professors R. D. James, Charles Loewner, R. S. Phillips, J. G. Wendel, and R. E. Bellman.

The Canadian Mathematical Congress met jointly with the Mathematical Association of America on Monday afternoon.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of: C. F. Barr, Chairman; H. M. Gehman, J. W. Green, Greta Neubauer, W. T. Puckett, O. H. Rechard, Nathan Schwid, S. R. Smith, W. N. Smith, P. O. Steen, V. J. Varineau.

Registration headquarters was in the reception room of Wyoming Hall of the University of Wyoming. Dormitory accommodations were available in Wyoming Hall and in Knight Hall from Sunday, August 29 until Saturday, Sep-

tember 4. Meals were served cafeteria style in the Knight Hall Cafeteria. A tea for the women attending the meeting was held on Monday afternoon in Knight Hall Banquet Room. On Monday evening movies showing scenes in Wyoming were shown in the Liberal Arts College Auditorium. On Tuesday afternoon a trip was conducted to Vedauwoo, a place of great natural beauty in the mountains east of Laramie.

On Wednesday afternoon there was a trip by bus and car to Snowy Range, lying forty miles west of Laramie, which was followed by a wild-game (deer, antelope, elk) steak fry at the University Recreation Camp, at the foot of Snowy Range.

A square dance was held on Thursday evening. A local square dance group gave several exhibition numbers during the evening.

At the steak fry on Wednesday afternoon, President George D. Humphrey of the University of Wyoming expressed the greetings of the University to the assembled mathematicians. Professor W. L. Duren responded for the mathematical organizations and presented a resolution of thanks which was adopted unanimously expressing appreciation to the officers of the University and to the members of the Mathematics Department for their efforts in making this meeting so successful and enjoyable.

HARRY M. GEHMAN, *Secretary-Treasurer*

THE MAY MEETING OF THE MINNESOTA SECTION

The May meeting of the Minnesota Section of the Mathematical Association of America was held at Hamline University in St. Paul, Minnesota, on May 8, 1954. Sessions were held in the forenoon, at luncheon and in the afternoon. Professors K. H. Bracewell, G. K. Kalisch and A. G. Hill, Chairman of the Section, presided at the respective sessions.

Seventy-one persons attended the meeting including the following fifty-eight members of the Association:

N. E. Albrecht, H. M. Anderson, Souren Babikian, E. J. Berger, H. D. Block, C. R. Bonnell, K. H. Bracewell, G. U. Brauer, R. W. Brink, W. H. Bussey, Anne W. Calloway, J. M. Calloway, E. J. Camp, C. S. Carlson, Elizabeth Carlson, H. D. Colson, W. L. Duren, Jr., Helen Engebretson, I. C. Fischer, Walter Fleming, Abraham Franck, Gladys Gibbens, Ruby M. Grimes, K. L. Hankerson, Charles Hatfield, F. C. Hatfield, A. G. Hill, Hildegarde H. Howden, R. T. John, D. A. Johnson, G. K. Kalisch, W. S. Loud, W. J. Lyche, H. B. MacDougal, K. O. May, W. H. McBride, W. R. McEwen, E. O. Nelson, J. M. H. Olmsted, J. C. Peterson, Ruth Scholten, Sister M. Bibiana, Sister M. Joanne, Sister Mary Leontius, F. C. Smith, Marion V. Smith, O. E. Stanaitis, A. G. Swanson, F. J. Taylor, Takashi Terami, Matilda B. Thompson, H. L. Turrittin, Frances E. Walsh, K. W. Wegner, Irene L. Wente, R. P. Winter, F. L. Wolf, L. G. Woodby.

At the business meeting, the following officers were elected for the coming year: Chairman, Sister Mary Leontius, College of St. Teresa; Secretary, Professor F. C. Smith, College of St. Thomas; Executive Committee, Professor Walter Fleming, Mankato State Teachers College, Professor A. G. Hill, North

Dakota Agricultural College, Professor Fulton Koehler, University of Minnesota.

By invitation of the Executive Committee, Professor W. L. Duren, Jr., of Tulane University delivered an address at the morning session entitled "Boundary Conditions for the Undergraduate Program in Mathematics."

The following short papers were presented:

1. *The game tree for a simplified form of chess*, by Professor F. L. Wolf, Carleton College.

A simplified form of chess which uses a board of only nine squares and has each player starting with just a king and a queen was considered. A game tree was constructed for the game and optimal strategies were immediately found from the tree. Even without an effective stop rule to limit the number of moves, it is possible to show all essentials of the tree by making use of the fact that the physical position of the board is repeated in any cyclic sequence of moves. The optimal strategies for chess itself can be found (theoretically) in a similar manner.

2. *The tangential approach to area integration*, by Mr. C. R. Bonnell, Cedar Engineering Corporation.

The area bounded by a curve $y=f(x)$ and the x-axis in the interval $a \leqq x \leqq b$ can be computed by the following formula:

$$\int_a^b f(x)dx = \frac{1}{2}\frac{[f(b)]^2}{f'(b)} - \frac{1}{2}\frac{[f(a)]^2}{f'(a)} + \frac{1}{2}\int_a^b \frac{[f(x)]^2 f''(x)}{[f'(x)]^2}dx.$$

The development of the area equation indicates that another rule for computing areas using tangents and subtangents can be added to the list with Simpson's rule.

A special case of interest is $f(x)=ax^2$ because the two right triangles at the end points have a difference in area of 3/4 that of the area bounded.

3. *Calculating machines and mathematicians as they seemed to Oliver Wendell Holmes*, by Professor W. H. Bussey, University of Minnesota.

Professor Bussey read the following quotation from *The Autocrat of the Breakfast Table* and made some remarks about it: "Given certain factors, a sound brain should always evolve the same fixed product with the certainty of Babbage's calculating machine. What a satire, by the way, is that machine on the mere mathematician! A Frankenstein monster, a thing without brains and without heart, too stupid to make a blunder; that turns out results like a corn sheller, and never grows any wiser or better though it grind a thousand bushels of them!

"I have an immense respect for a man of talents *plus* 'the mathematics.' But the calculating power alone should seem to be the least human of qualities and to have the smallest amount of reason in it; since a machine can be made to do the work of three or four calculators and better than any of them. Sometimes I have been troubled that I had not a deeper intuitive apprehension of the relations of numbers. But the triumph of the ciphering hand-organ has consoled me. I always fancy I can hear the wheels clicking in a calculator's brain."

4. *A theorem on transcendentality*, by Mr. Richard Juberg, University of Minnesota, introduced by Professor S. E. Warschawski.

Define $p^{(n)}=p(p-1)\cdots(p-n+1)$; $p^{(0)}=1$. Let a be algebraic and non-rational; let $b=ac+d$ where c and d are rational.

Theorem: The analytic function $f_\lambda(z)$, generated by the power series $\sum_{n=0}^{\infty}(an+b)^{(n+\lambda)}z^n/n!$, $(\lambda=0, 1, 2, \cdots)$, has the property that for all algebraic z except $z=0$, the corresponding functional values are all transcendental. An outline of the proof was given under the assumptions that

c and d were integers and that z was confined to the circle of convergence of the generating functional element.

5. *On convolutions in number theory*, by Professor Jan Popken, University of Utrecht and the University of Minnesota, introduced by Professor S. E. Warschawski.

Let $f(n)$ and $g(n)$ be two arithmetical functions. The new arithmetical function of n defined by $\sum_{d\delta=n} f(d)g(\delta)$ is called the convolution of $f(n)$ and $g(n)$. It is denoted by $f(n) * g(n)$. The commutative and associative laws hold.

Some proofs in the theory of numbers can be simplified considerably by the introduction of these convolutions. This was shown first for a simple example and then for the elementary proof of the prime number theorem.

6. *On the minimality of the variational principles of classical particle mechanics*, by Professor H. D. Block, University of Minnesota.

An elementary device, apparently overlooked in the standard techniques of the calculus of variations, is particularly effective in dealing with the variational integrals of mechanics and enables us to settle the following question.

Which of the variational principles are minimal principles (under suitable conditions), which of them are not minimal principles under any conditions and what are the "suitable conditions"?

F. C. SMITH, *Secretary*

THE MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The tenth annual meeting of the Upper New York State Section of the Mathematical Association of America was held at New York State College for Teachers at Albany, New York, on May 1, 1954. The Chairman of the Section, Professor Harriet F. Montague of the University of Buffalo, presided at the morning session, and the Vice-Chairman, Professor J. R. F. Kent of Harpur College, presided at the afternoon session.

Seventy-three persons attended the meeting, including the following fifty-three members of the Association:

E. B. Allen, M. R. Bates, R. A. Beaver, Col. W. W. Bessell, Harry Birchenough, E. A. Butler, Ethel B. Callahan, J. D. Campbell, A. J. Coleman, E. J. Downie, O. J. Farrell, A. H. Fox, N. S. Free, J. E. Freund, H. M. Gehman, Lillian Gough, N. G. Gunderson, A. S. Hendler, H. K. Holt, H. F. Hunter, J. R. F. Kent, D. E. Kibbey, Violet H. Larney, R. D. Larsson, Roger Lessard, Caroline A. Lester, J. V. Limpert, R. C. Luippold, Ingo Maddaus, Jr., Dis Maly, J. N. Mangnall, June M. McArtney, Myles McConnon, Norman Miller, Harriet F. Montague, Mabel D. Montgomery, D. S. Morse, C. W. Munshower, C. V. Newsom, F. D. Parker, M. J. Pascual, Valdemars Punga, Rev. Timothy Reardon, D. A. Robinson, H. M. Rosenbaum, Edith R. Schneckenburger, Sister Noel Marie, Ruth W. Stokes, W. C. Stone, R. L. Swain, Nura D. Turner, F. C. Warner, W. G. Warnock.

At the business meeting the following officers were elected: Chairman, Professor J. R. F. Kent, Harpur College; Vice-Chairman, Professor C. E. Rhodes, Alfred University; Secretary, Professor N. G. Gunderson, University of Rochester. A motion was passed which created an Executive Committee consisting of the officers of the Section and the Sectional Governor. The motion directed

the Executive Committee to prepare a revision of the By-Laws of the Section and to present the revision at the 1955 meeting for the consideration of the members.

The following papers were presented:

1. *A program for the preparation of secondary school teachers of mathematics*, by Professor R. A. Beaver, New York State College for Teachers at Albany.

The program for training junior and senior high school teachers of mathematics at the New York State College for Teachers at Albany was described. In terms of semester credit hours, the usual program consists of distributional requirements other than mathematics, 38 hours; professional courses, 18 hours; mathematics, 24 hours; a minor field, 24 hours; free electives, 20 hours.

2. *Potential of a charged cylinder between two parallel grounded planes*, by Dr. Hillel Poritsky, General Electric Company; read by Mr. R. A. Powell, General Electric Company.

A solution is obtained for the harmonic function $V(x, y)$ in the region outside a circle $r=a$, $a<\pi/2$, and between the lines $x=\pm\pi/2$, subject to the boundary conditions $V=1$ on $r=a$, $V=0$ on $x=\pm\pi/2$.

The conditions $V=0$ on $x=\pm\pi/2$ are satisfied by putting $V=\text{Re}\,[f(z)]$, $z=x+iy$, and letting $f(z)=\sum C_n F^{(n)}(z)$, where $F=-\ln\tan(z/2)$, C_n are real, and n is even. By expanding $F+\ln(z/2)$ in powers of z, differentiating to obtain similar expansions for $F^{(n)}$, and applying the condition $V=1$ on $r=a$, one obtains an infinite set of linear equations in C_n. These are solved by an iteration method based on the observation that the main diagonal coefficients are large compared to the remaining coefficients.

3. *Statistics and engineering*, by Professor Roger Lessard, École Polytechnique, Montreal.

The speaker reviewed the developments of the engineering applications of statistics. By several examples, he showed why numerous societies have recommended the introduction of statistical courses in all engineering curricula, and gave the different ways used by universities to solve this problem. He also recommended that the statistical concept be introduced as soon as possible in high schools.

4. *The 1953 Summer Conference in Collegiate Mathematics*, by Professor Ethel B. Callahan, Hartwick College.

The speaker described the objectives and organization of the Colorado Conference, and reported on the courses and other activities offered.

5. *Representations of compact Lie groups*, by Professor A. J. Coleman, University of Toronto.

The classification of simple compact Lie groups and their representations, originally due to Cartan, is presented using the in-the-large methods of H. Weyl and H. Hopf. The Pontrjagin duality theory permits the association with the group G of an n-dimensional lattice, characterizing G, if G has rank n. All possible lattices arising in this way have been classified by Coxeter [*Can. J. of Math.* vol. 3, 1951, pp. 391–441] who associates with each lattice a connected graph. The use of the Coxeter graph in studying the properties of a representation is illustrated by a simple solution of the problem of determining the number of homologues of any weight in a representation. The following inclusion relations amongst the exceptional Lie groups is pointed out: $G_2 \subset F_4 \subset E_6 \subset E_7 \subset E_8$.

6. *Mathematics as taught to engineers by Dr. Steinmetz*, by Mr. P. L. Alger, General Electric Company, introduced by Miss Nura D. Turner.

Dr. Charles P. Steinmetz was a great teacher, who did far more than any one else to educate electrical engineers in the then new science of alternating currents. His 300-page book *Engineering Mathematics* (1911) begins with arithmetic and goes through trigonometry, differential equations, and the theory of functions. He made mathematics appear as a simple well-knit whole, and all its processes appear to be merely extensions of arithmetic. By using general numbers, $a+jb$, throughout, instead of only real numbers, he telescoped all kinds of mathematics into simple and usable forms, especially useful to engineers.

7. *A non-linear problem from nuclear reactor theory*, by Professor A. H. Fox, Union College.

Consideration of the relation between power and temperature in a nuclear reactor with a negative temperature coefficient of reactivity leads to a non-linear differential equation for the transient temperature rise from steady state operation. A study of this equation determines the stability of the system for typical values of the parameters involved.

8. *Modern differential geometry and its application to relativity*, by Professor Valdemars Punga, Rensselaer Polytechnic Institute.

The speaker discussed the growth of the concepts of differential geometry from Riemannian space to Finsler space and to the affinely connected manifolds, and the associated development of relativity. He included a discussion of the principle of parallel displacement of tensors defined by affine connection $\Gamma^{\alpha}_{\beta\gamma}$ and its possible generalizations.

N. G. GUNDERSON, *Secretary*

THE JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The eighth annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at Reed College, Portland, Oregon, on June 18, 1954, in conjunction with the five hundred fourth meeting of the American Mathematical Society. Professors R. D. James and Ivan Niven were invited by Professor Harold Chatland, Chairman of the Section, to share the duties of presiding at the afternoon session.

Seventy-three persons were in attendance, including the following fifty-five members of the Association:

C. B. Allendoerfer, J. J. Andrews, T. M. Apostol, M. G. Arsove, R. W. Ball, R. A. Beaumont, R. F. Bell, J. L. Botsford, J. L. Brenner, L. G. Butler, F. M. Carpenter, Harold Chatland, P. A. Clement, K. L. Cooke, C. M. Cramlet, D. B. Dekker, K. S. Ghent, W. M. Gilbert, S. G. Hacker, Mary E. Haller, Burrowes Hunt, H. H. Irwin, R. D. James, J. M. Kingston, M. S. Knebelman, D. H. Lehmer, R. B. Leipnik, A. E. Livingston, R. G. Long, A. T. Lonseth, C. F. Luther, J. H. McKay, A. F. Moursund, B. N. Moyls, D. C. Murdoch, W. M. Myers, Jr., J. A. Nickel, Ivan Niven, C. O. Oakley, Gloria Olive, T. G. Ostrom, T. S. Peterson, Ruth E. Porter, C. A. Pursel, L. B. Rall, J. B. Roberts, E. M. Scheuer, A. J. Smith, W. M. Stone, D. B. Tillotson, J. R. Vatnsdal, Sylvia Vopni, L. B. Williams, R. M. Winger, F. H. Young.

Following a joint dinner with the American Mathematical Society, a business meeting was held in the evening at which the following officers were elected: Chairman, Professor Ivan Niven, University of Oregon; Vice-Chairman, Professor D. C. Murdoch, University of British Columbia; Secretary-Treasurer, Professor J. M. Kingston, University of Washington. The by-laws of the Section were amended to provide for an Executive Committee consisting of the officers and the Governor of the Section. Reports of contests held in British Columbia, Washington and Oregon were received. A Program Committee for the 1955 meeting was appointed, consisting of Professor D. C. Murdoch, Chairman, and Professors J. L. Botsford, and W. M. Myers, Jr.

The afternoon session consisted of the following invited hour address, two fifteen-minute papers and a symposium by a panel of four speakers:

1. Invited address: *Modern mathematics for freshmen*, by Professor C. O. Oakley, Haverford College.

The standard undergraduate curriculum in mathematics, especially that of the freshman year consisting of trigonometry, algebra and analytic geometry, is obsolete. In order to keep the youth of the nation abreast of the current developments in mathematics, in the physical sciences and in the social sciences, it is essential that modern mathematical ideas be introduced at a very early date—preferably in the high school. A detailed curriculum for college freshmen was suggested including logic, sets, groups, fields, functions (ordered pairs), limits, polynomial calculus, probability and statistics *and enough* trigonometry, algebra and analytic geometry for a following and substantial course in calculus.

2. *The most desirable undergraduate preparation for advanced study in mathematics*, by Professor D. B. Tillotson, Northwest Nazarene College.

A survey was made of the opinions of the chairmen of the mathematics departments of leading graduate schools concerning a program of fifteen semester hours of advanced undergraduate mathematics which would best prepare a student in a liberal arts college to pursue graduate work. The replies seem to indicate that about six hours of advanced algebra and at least six hours of advanced calculus including differential equations are very desirable. The remaining three hours were generally given either to geometry or to more analysis.

3. *On simplification of certain probability density functions*, by Professor W. M. Stone, Boeing Airplane Company and Oregon State College.

A number of interesting problems in communication theory involve the distribution of a constant amplitude sinusoidal signal in the presence of normal background noise. It seems that much greater simplicity is obtained if a more realistic randomly modulated signal amplitude is postulated. For example, the distribution of current in the presence of normal noise,

$$f(x) = \pi^{-1} \int_0^{\pi} \phi(x - y \cos \theta) d\theta$$

where $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ and y is constant, reduces to a simple normal law if y is assumed to be Rayleigh distributed.

4. The Symposium: *The content of undergraduate mathematics courses*, moderated by Professor Ivan Niven, University of Oregon.

Panel:

(1) *College mathematics for engineers and scientists*, by Professor D. R. Crosby, University of Alberta.

The speaker expressed his opposition to the introduction into universities of the course recently proposed by Professors Allendoerfer and Oakley.* His opposition was based on the low entrance requirement for the course. In the preface it is stated that "A course based on (the first five chapters) is now given at the University of Washington with only two years of (United States) high school mathematics as a prerequisite. . . . After Chapters VI to X (which require intermediate algebra as a prerequisite) a student is prepared to *BEGIN* a standard course in calculus." For engineers and scientists, beginning calculus in second year University is far too late. The speaker did not believe that the "democratic" United States high schools would improve or even maintain their already depleted mathematics courses for the benefit of students of science, if the United States universities established new lows in entrance requirements for the large mass of students. Although sympathetic to introducing modern mathematics earlier, the speaker thought that the Allendoerfer-Oakley course would do more harm than good.

(2) *The content of calculus courses*, by Professor R. D. James, University of British Columbia.

Professor Menger, in the preface to his book, *Calculus, A Modern Approach*, speaks of "a project to rescue calculus from one of the greatest dangers that may befall a discipline—the danger of petrifaction." This paper suggests ways and means of avoiding the danger.

(3) *Suggested revisions in college mathematics*, by Professor R. B. Leipnik, University of Washington.

The traditional undergraduate curriculum, stemming from Euler's classic text of 1748, is admirably shaped for application to Newtonian mechanics. However, notions of logic, set algebra, probability, and vector space are now needed by students of electrical engineering, physics, and quantitative social and biological science. Experience with students at the University of Washington indicates that some of these topics can be successfully introduced much earlier than is usually believed possible.

(4) *Some practical aspects of the teaching of modern mathematics to undergraduates* by Professor M. G. Arsove, University of Washington.

It is both desirable and inevitable that "conventional" undergraduate mathematics courses give way to courses containing more modern abstract material. Practical aspects of this transition require the consideration of problems in course content, pedagogy, and administration. Course content will depend partly on circumstances at individual institutions. However, it is important to discuss mathematical logic and the structure of a mathematical system, so that the rules are known before the game begins. From the pedagogical viewpoint it is desirable that the abstract systems considered be limited in number and well delineated. The administrative problems arising from curriculum revision are perhaps the most troublesome of the problems encountered.

J. MAURICE KINGSTON, *Secretary*

* Principles of Mathematics, by Carl B. Allendoerfer and Cletus O. Oakley, published in mimeographed form by McGraw-Hill Book Co., 1953.

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CALENDAR OF FUTURE MEETINGS

Thirty-eighth Annual Meeting, University of Pittsburgh, Pittsburgh, Pennsylvania, December 30, 1954.

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pennsylvania, May, 1955.

ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.

INDIANA, Butler University, Indianapolis, May, 1955.

IOWA, St. Ambrose College, Davenport, April 15–16, 1955.

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 18–19, 1955.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Georgetown University, Washington, D. C., December 4, 1954.

METROPOLITAN NEW YORK, Queens College, Flushing, New York, April 30, 1955.

MICHIGAN, Michigan State College, East Lansing, Spring, 1955.

MINNESOTA, College of St. Teresa, Winona, Minnesota, May, 1955.

MISSOURI, University of Kansas City, Spring, 1955.

NEBRASKA

NORTHERN CALIFORNIA, University of California, Berkeley, January 15, 1955.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 17, 1955.

PHILADELPHIA, Princeton University, Princeton, New Jersey, November 27, 1954.

ROCKY MOUNTAIN, University of Wyoming, Laramie, Spring, 1955.

SOUTHEASTERN, Tennessee Polytechnic Institute, Cookeville, March 11–12, 1955.

SOUTHERN CALIFORNIA, Santa Monica City College, March 12, 1955.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1955.

TEXAS, Abilene Christian College, Abilene, April, 1955.

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